Chapter :- 01

Successive Differentiation & Leibnitz's Theoram

Successive Differentiation

If y be any function of x then its derivative dy/dx will be a function of x differentiation can be done Hence we can find again differential coefficient of dy/dx. The derivative of dy/dx i.e $\frac{d}{dx}\left[\frac{dy}{dx}\right]$ is called second order derivative of y

The successive derivative of y w.r. to x are derived by $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$

$$\Rightarrow \frac{dy}{dx} = y' = y_1 = f'(x) = D f(x) = Dy$$

$$\Rightarrow \frac{d^2y}{dx^2} = y'' = y_2 = f''(x) = D^2 f(x) = D^2 y$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\Rightarrow \frac{d^ny}{dx^n} = y^n = y_n = f^n(x) = D^n(x) = D^n y$$

Illustrative Examples

Q1) If $y = \sin(m \sin^{-1}x)$ then prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$$

Sol. $Y = \sin(m\sin^{-1}x)$

$$\frac{dY}{dx} = \cos(m\sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dY}{dx} = m\cos(m\sin^{-1}x)$$

Squaring both side

$$\Rightarrow (1-x^2)\left[\frac{dY}{dx}\right]^2 = m^2\cos^2(m\sin^{-1}x) \quad \left\{\because \cos^2 x = 1-\sin^2 x\right\}$$

$$\Rightarrow (1-x^2)\left[\frac{dY}{dx}\right]^2 = m^2\left[1-\sin^2(m\sin^{-1}x)\right]$$

$$\{\because Y = \sin(m(\sin^{-1}x))\}$$

$$\Rightarrow (1-x^2)\left[\frac{dY}{dx}\right]^2 = m^2\left[1-Y^2\right]$$

again diff. both side w.r. to x

By Product rule

$$\Rightarrow (1-x^2)\,2\frac{dY}{dx}\cdot\frac{d^2Y}{dx^2} - 2x\left(\frac{dY}{dx}\right)^2 = m^2\left(0-2Y\frac{dY}{dx}\right)$$

$$\Rightarrow 2(1-x^2)\frac{dY}{dx}\cdot\frac{d^2Y}{dx^2} - 2x\left[\frac{dY}{dx}\right]^2 + 2Ym^2\frac{dY}{dx}$$

$$\Rightarrow 2\frac{dY}{dx}\left[(1-x^2)\frac{d^2Y}{dx^2} - x\frac{dY}{dx} + m^2Y\right] = 0$$

$$\Rightarrow \boxed{(1-x^2)\frac{d^2Y}{dx^2} - x\frac{dY}{dx} + m^2Y = 0}$$

Hence Proved

Q2) If $P^2 = a^2\cos^2\theta + b^2\sin^2\theta$ the Prove that $P + \frac{d^2Y}{} = \frac{a^2b^2}{}$

Sol. $P^2 = a^2\cos^2\theta + b^2\sin^2\theta$

$$\Rightarrow 2P\frac{dP}{d\theta} = a^2(-2\cos\theta\sin\theta) + b^2(2\sin\theta\cos\theta)$$

$$\Rightarrow 2P\frac{dP}{d\theta} = -a^2\sin 2\theta + b^2\sin 2\theta$$

$$\Rightarrow 2P\frac{dP}{d\theta} = (b^2-a^2)\sin 2\theta$$

again diff both side w.r. to 'θ'

$$\Rightarrow 2P\frac{d^2P}{d\theta^2} + 2\frac{dP}{d\theta}\frac{dP}{d\theta} = (b^2-a^2)2\cos 2\theta$$

$$\Rightarrow P\frac{d^2P}{d\theta^2} + \left[\frac{dP}{d\theta}\right]^2 = (b^2-a^2)\cos 2\theta$$

multiply by P^2 on both side

$$\Rightarrow P^3\frac{d^2P}{d\theta^2} = P^2(b^2-a^2)\cos 2\theta - P^2\left(\frac{dP}{d\theta}\right)^2$$

adding P^4 on both side

$$\Rightarrow P^4 + P^3\frac{d^2P}{d\theta^2} = P^4 + P^2(b^2-a^2)\cos 2\theta - P^4\left(\frac{dP}{d\theta}\right)^2$$

$$\Rightarrow P^4 + P^3\frac{d^2P}{d\theta^2} = P^2\left[P^2 + (b^2-a^2)\cos 2\theta - P^2\left(\frac{dP}{d\theta}\right)^2\right]$$

$$\{\because P^2 = a^2\cos^2\theta + b^2\sin^2\theta\}$$

$$\{\because \cos 2\theta = \cos^2\theta - \sin^2\theta\}$$

$$\Rightarrow P^4 + P^3\frac{d^2P}{d\theta^2} = (a^2\cos^2\theta + b^2\sin^2\theta)\left[a^2\cos^2\theta + b^2\sin^2\theta + (b^2-a^2)(\cos^2\theta - \sin^2\theta) - (b^2-a^2)^2\sin^2\cos^2\theta\right]$$

$$\Rightarrow P^4 + P^3\frac{d^2P}{d\theta^2} = a^2b^2\left[\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\right]$$

$\Rightarrow p^4+p^3\frac{d^2p}{d\theta^2} = a^2b^2(\cos^2\theta+\sin^2\theta)^2$

$\Rightarrow p^4+p^3\frac{d^2p}{d\theta^2} = a^2b^2$ Dividing by p^3 both side

$\Rightarrow \boxed{p+\frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}}$ Hence Proved

Q3) If $y = A\sin mx + B\cos mx$, then prove that $\frac{d^2y}{dx^2}+m^2y=0$

Sol^n $y = A\sin mx + B\cos mx$

$\Rightarrow \frac{dy}{dx} = Am\cos mx - Bm\sin mx$

again diff. w. r. to x

$\Rightarrow \frac{d^2y}{dx^2} = -Am^2\sin mx - Bm^2\cos mx$

$\Rightarrow \frac{d^2y}{dx^2} = -m^2(A\sin mx + B\cos mx)$

$\Rightarrow \boxed{\frac{d^2y}{dx^2}+m^2y=0}$ Hence Proved.

N^{th} derivative of some standard function

- $D^n(e^{ax+b}) = a^n e^{ax+b}$
- $D^n(a^x) = (\log a)^n a^x$
- $D^n(ax+b)^m = m(m-1)(m-2)\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}$ $\{n<m\}$

• Particular case:-

① If $n>m$, $D^n(ax+b)^m = 0$

② If $m=n$, $D^n(ax+b)^n = n!\,a^n$

③ If $m=-1$, $D^n(ax+b)^{-1} = (-1)^n n!\,a^n(ax+b)^{-n-1}$

$D^n(ax+b)^{-1} = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$

- $D^n\sin(ax+b) = a^n\sin(ax+b+1/2\cdot n\pi)$
- $D^n\cos(ax+b) = a^n\cos(ax+b+1/2\cdot n\pi)$

Q1) find the n^{th} derivative of $\frac{x^3}{x^2-3x+2}$

Sol^n $\frac{x^3}{x^2-3x+2} = x+3+\frac{7x-6}{x^2-3x+2}$

$\Rightarrow \frac{x^3}{x^2-3x+2} = x+3-\frac{1}{x-1}+\frac{8}{x-2}$

$\Rightarrow D^n\left[\frac{x^3}{x^2-3x+2}\right] = D^n(x)+D^n(3) - D^n[(x-1)^{-1}] + 8\,D^n[(x-2)^{-1}]$

$\Rightarrow D^n\left[\frac{x^3}{x^2-3x+2}\right] = 0+0-\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{8(-1)^n n!}{(x-2)^{n+1}}$

$\Rightarrow D^n\left[\frac{x^3}{x^2-3x+2}\right] = (-1)^n n!\left[\frac{8}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$ ans

Rough work

$$\begin{array}{r|l} x^2-3x+2 & x^3 \\ & -(x^3-3x^2+2x) \\ & 3x^2-2x \\ & -(3x^2-9x+6) \\ & 7x-6 \end{array} \quad x+3$$

$\frac{7x-6}{(x-1)(x-2)} = \frac{A}{(x-1)}+\frac{B}{(x-2)}$

Put $x=1$

$A = \frac{7-6}{-1} = -1$

Put $x=2$

$B = \frac{14-6}{1} = 8$

Q2) find the n^{th} derivative of coefficient of $1/x^2-a^2$

Sol^n

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$$\Rightarrow \frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{(x-a)} + \frac{B}{(x+a)}$$

Put $x = a$, $A = 1/2a$

Put $x = -a$, $B = -1/2a$

$$\Rightarrow D^n\left[\frac{1}{x^2-a^2}\right] = \frac{1}{2a} D^n\left[\frac{1}{x-a}\right] - \frac{1}{2a} D^n\left[\frac{1}{x+a}\right]$$

$$\Rightarrow D^n\left[\frac{1}{x^2-a^2}\right] = \frac{1}{2a} D^n (x-a)^{-1} - \frac{1}{2a} D^n (x+a)^{-1}$$

$$\left\{ \because D^n (ax+b)^{-1} = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}} \right\}$$

$$\Rightarrow D^n\left[\frac{1}{x^2-a^2}\right] = \frac{1}{2a}\left[\frac{(-1)^n n!}{(x-a)^{n+1}}\right] - \frac{1}{2a}\left[\frac{(-1)^n n!}{(x+a)^{n+1}}\right]$$

$$\Rightarrow D^n\left[\frac{1}{x^2-a^2}\right] = \frac{(-1)^n n!}{2a}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right]$$

Leibnitz's Theoram :- The n^{th} differential coefficient of the product of two function is conviniently evaluated by use of Leibnitz's Theoram

Leibnitz's Theoram says If u & v are two functions of x then

$$D^n(uv) = D^n(u)\cdot v + {}^nC_1 D^{n-1}(u)\, D(v) + {}^nC_2 D^{n-2}(u)\, D^2(v) + \dots + {}^nC_n (u) D^n(v)$$

Q1) find the n^{th} differential coefficient of $x^3 \cos x$

Sol^n $\left\{ D^n(\cos x) = \cos\left(x + \frac{n\pi}{2}\right) \right\}$

$$\Rightarrow D^n(x^3\cos x) = D^n(\cos x)\cdot x^3 + {}^nC_1 D^{n-1}(\cos x)\, D(x^3) + {}^nC_2 D^{n-2}(\cos x)\, D^2(x^3) + {}^nC_3 D^{n-3}(\cos x)\, D^3(x^3) + {}^nC_4 D^{n-4}(\cos x)\, D^4(x^3) + \dots$$

$$\Rightarrow D^n(x^3\cos x) = x^3\cos\left[x+\frac{n\pi}{2}\right] + n(3x^2)\cos\left[x+\frac{(n-1)\pi}{2}\right] + \frac{n(n-1)}{1\times\cancel{2}}(\cancel{6}^{3}x)\cos\left[x+\frac{(n-2)\pi}{2}\right] + \frac{n(n-1)(n-2)(\cancel{6})}{1\times\cancel{2}\times\cancel{3}}\cos\left[x+\frac{(n-3)\pi}{2}\right]$$

$$\left\{ \because \cos\left[x+\frac{(n-1)\pi}{2}\right] = \cos\left[x+\frac{n\pi}{2}-\frac{\pi}{2}\right] \right.$$
$$\left. \cos\left[x+\frac{(n-1)\pi}{2}\right] = \sin\left[x+\frac{n\pi}{2}\right] \right\}$$

$$\left\{ \because \cos\left[x+\frac{(n-2)\pi}{2}\right] = \cos\left[x+\frac{n\pi}{2}-\pi\right] \right.$$
$$\left. \Rightarrow \cos\left[x+\frac{(n-2)\pi}{2}\right] = \cos\left[\pi-\left(x+\frac{n\pi}{2}\right)\right] = -\cos\left[x+\frac{n\pi}{2}\right] \right\}$$

$$\left\{ \because \cos\left[x+\frac{(n-3)\pi}{2}\right] = \cos\left[x+\frac{n\pi}{2}-\frac{3\pi}{2}\right] = \cos\left[\frac{3\pi}{2}-\left(x+\frac{n\pi}{2}\right)\right] = -\sin\left[x+\frac{n\pi}{2}\right] \right\}$$

$$\Rightarrow D^n(x^3\cos x) = x^3\cos\left[x+\frac{n\pi}{2}\right] + 3x^2 n\sin\left[x+\frac{n\pi}{2}\right] \mp 3x[n(n-1)]\cos\left[x+\frac{n\pi}{2}\right] - n(n-1)(n-2)\sin\left[x+\frac{n\pi}{2}\right]$$

$\Rightarrow D^n(x^3 \cos x) = x\left[x^2 - 3n(n-1)\right]\cos\left[x + \frac{n\pi}{2}\right] + n\left[6x^2 - (n-1)(n-2)\right]\sin\left[x + \frac{n\pi}{2}\right]$ ans

Q2) If $Y = (\sin^{-1}x)^2$ then prove that, $(1-x^2)Y_{n+2} - (2n+1)xY_{n+1} - n^2 Y_n = 0$

Sol^n: $Y = (\sin^{-1}x)^2$

$Y_1 = 2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, Y_1 = 2(\sin^{-1}x)$

squaring both side we get,

$\Rightarrow (1-x^2)Y_1^2 = 4(\sin^{-1}x)^2$

$\Rightarrow (1-x^2)Y_1^2 = 4Y$ (again diff.)

$\Rightarrow (1-x^2)2Y_1Y_2 + Y_1^2(-2x) = 4Y_1$

$\Rightarrow (1-x^2)2Y_1Y_2 - 2xY_1^2 - 4Y_1 = 0$

$\Rightarrow 2Y_1\left[(1-x^2)Y_2 - xY_1 - 2\right] = 0$

$\Rightarrow (1-x^2)Y_2 - xY_1 - 2 = 0$

diff. n times by Leibnitz's theorem

$\Rightarrow D^n(1-x^2)Y_2 - D^n(xY_1) - D^n(2) = 0$

$\Rightarrow \left[D^n(Y_2)(1-x^2) + {}^nC_1 D^{n-1}(Y_2)D(1-x^2) + {}^nC_2 D^{n-2}(Y_2) D^2(1-x^2)\right] - \left[D^n(Y_1)x + {}^nC_1 D^{n-1}(Y_1)D(x)\right] - 0 = 0$

$\Rightarrow (1-x^2)Y_{n+2} + nY_{n+1}(-2x) + \frac{n(n-1)}{1\times 2}Y_n(-2)$

$-\left[xY_{n+1} + nY_n\right] = 0$

$\Rightarrow (1-x^2)Y_{n+2} - (2n+1)xY_{n+1} - \left[n^2 - n + n\right]Y_n = 0$

$\Rightarrow (1-x^2)Y_{n+2} - (2n+1)xY_{n+1} - n^2 Y_n = 0$

Q3) If $Y = \sin(m\sin^{-1}x)$ then Prove that $(1-x^2)Y_2 - xY_1 + m^2Y = 0$ and deduce it $(1-x^2)Y_{n+2} - (2n+1)xY_{n+1} - (n^2 - m^2)Y_n = 0$

Sol^n $Y = \sin(m\sin^{-1}x)$

$\Rightarrow Y_1 = \cos(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, Y_1 = m\cos(m\sin^{-1}x)$

squaring both side we get

$\Rightarrow (1-x^2)Y_1^2 = m^2\cos^2(m\sin^{-1}x)$

$\Rightarrow (1-x^2)Y_1^2 = m^2\left(1 - \sin^2(m\sin^{-1}x)\right)$

$\Rightarrow (1-x^2)Y_1^2 = m^2(1 - Y^2)$

again diff. w. r. to 'x'

$\Rightarrow (1-x^2)2Y_1Y_2 + Y_1^2(-2x) = m^2(-2YY_1)$

$\Rightarrow (1-x^2)2Y_1Y_2 - 2xY_1^2 + 2YY_1m^2 = 0$

dividing by $2Y_1$

$\Rightarrow (1-x^2)Y_2 - xY_1 + m^2Y = 0$

diff. n times by Leibnitz's theorem

$\Rightarrow D^n(1-x^2)Y_2 - D^n(xY_1) + D^n(m^2Y) = 0$

$\Rightarrow \left[D^n(Y_2)(1-x^2) + {}^nC_1 D^{n-1}(Y_2)D(1-x^2) + {}^nC_2 D^{n-2}(Y_2)D^2(1-x^2)\right] - \left[D^n(Y_1)x + {}^nC_1 D^{n-1}(Y_1)D(x)\right] + m^2 D^n(Y) = 0$

$\Rightarrow (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1\times 2}(-2)y_n$

$- xy_{n+1} - ny_n + m^2 y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2 - \cancel{n} + \cancel{n} - m^2)y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

Q4) If $y = a\cos(\log x) + b\sin(\log x)$ then PT
$x^2y_2 + xy_1 + y = 0$ &
$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Sol^n $y = a\cos(\log x) + b\sin(\log x)$
$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$

$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$
again diff. w. r. to x

$\Rightarrow xy_2 + y_1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$

$\Rightarrow x^2y_2 + xy_1 = -(a\cos(\log x) + b\sin(\log x))$
$\Rightarrow x^2y_2 + xy_1 = -y \Rightarrow x^2y_2 + xy_1 + y = 0$
diff. n times by Leibnitz's theoram
$\Rightarrow D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$
$\Rightarrow [D^n(y_2)x^2 + {}^nC_1 D^{n-1}(y_2)D(x^2) + {}^nC_2 D^{n-2}(y_2)D^2(x^2)]$
$+ [D^n(y_1)x + {}^nC_1 D^{n-1}(y_1)D(x)] + D^n(y) = 0$

$\Rightarrow \left[x^2y_{n+2} + 2xny_{n+1} + \frac{n(n-1)}{1\times 2}(2)y_n\right] + [xy_{n+1}$
$+ ny_n] + y_n = 0$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2 - \cancel{n} + \cancel{n} + 1)y_n = 0$
$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1}\,(n^2+1)y_n = 0$

Q5) If $\cos^{-1}\left[\frac{y}{b}\right] = \log\left[\frac{x}{n}\right]^n$ then Prove that
$x^2y_2 + xy_1 + n^2y = 0$ &
$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

Sol^n $\cos^{-1}\left[\frac{y}{b}\right] = \log\left[\frac{x}{n}\right]^n \Rightarrow \frac{y}{b} = \cos\left[n\log\left(\frac{x}{n}\right)\right]$

$\Rightarrow y = b\cos\left[n\log\left[\frac{x}{n}\right]\right]$ diff. w. r. to 'x'

$\Rightarrow y_1 = -b\sin\left[n\log\left[\frac{x}{n}\right]\right]\cdot \cancel{n}\frac{1}{x/n}\frac{1}{\cancel{n}}$

$\Rightarrow xy_1 = -nb\sin[n\log(x/n)]$
again diff. w. r. to 'x'
$\Rightarrow xy_2 + y_1 = -nb\cos(n\log(x/n))\,\cancel{n}\frac{1}{x/n}\frac{1}{\cancel{n}}$

$\Rightarrow x^2y_2 + xy_1 = -n^2b\cos(n\log x/n)$
$\Rightarrow x^2y_2 + xy_1 = -n^2y \Rightarrow x^2y_2 + xy_1 + n^2y = 0$
diff. n times by Leibnitz's theoram
$\Rightarrow D^n(x^2y_2) + D^n(xy_1) + D^n(n^2y) = 0$
$\Rightarrow [D^n(y_2)x^2 + {}^nC_1 D^{n-1}(y_2)D(x^2) + {}^nC_2 D^{n-2}(y_2)D^2(x^2)]$
$+ [D^n(y_1)x + {}^nC_1 D^{n-1}(y_1)D(x)] + n^2D^n(y) = 0$

$\Rightarrow \left[x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{1\times 2}2y_n\right] + [xy_{n+1}$
$+ ny_n] + n^2y_n = 0$
$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2 - \cancel{n} + \cancel{n} + n^2)y_n$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$$

Q7) If $y^{1/m} + y^{-1/m} = 2x$ then Prove that

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Sol^n $y^{1/m} + \frac{1}{y^{1/m}} = 2x$

$$\Rightarrow (y^{1/m})^2 + 1 = 2xy^{1/m}$$

$$\Rightarrow (y^{1/m})^2 + 1 - 2xy^{1/m} = 0$$

$$\Rightarrow y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\left\{ \because D = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \right\}$$

$\Rightarrow y^{1/m} = x \pm \sqrt{x^2-1}$ applying Power of m BS

$$\Rightarrow y = (x + \sqrt{x^2-1})^m$$

diff. w. r. to. 'x'

$$\Rightarrow y_1 = m(x \pm \sqrt{x^2-1})^{m-1}\left[1 \pm \frac{1}{\cancel{2}\sqrt{x^2-1}}\cancel{2}x\right]$$

$$\Rightarrow y_1 = \frac{m(x \pm \sqrt{x^2-1})^{m-1}}{\sqrt{x^2-1}}\left[\sqrt{x^2-1} \pm x\right]$$

$\Rightarrow \sqrt{x^2-1}\,y_1 = m(x \pm \sqrt{x^2-1})^m$ Squaring BS

$\Rightarrow (x^2-1)y_1^2 = m^2y^2$ $\quad (\because y = (x \pm \sqrt{x^2-1})^m)$

again diff. w. r. to x

$$\Rightarrow (x^2-1)2y_1y_2 + 2xy_1^2 = m^2 2yy_1$$

$$\Rightarrow (x^2-1)2y_1y_2 + 2xy_1^2 - m^2 2yy_1 = 0$$

$$\Rightarrow 2y_1[(x^2-1)y_2 + xy_1 - m^2y] = 0$$

$$\Rightarrow (x^2-1)y_2 + xy_1 - m^2y = 0$$

diff. n times by Leibnitz's Theorem

$$\Rightarrow D^n[(x^2-1)y_2] + D^n(xy_1) - D^n(m^2y) = 0$$

$$\Rightarrow [D^n(y_2)(x^2-1) + {}^nC_1D^{n-1}(y_2)D(x^2-1) + {}^nC_2D^{n-2}(y_2)D^2(x^2-1)] + [D^n(y_1)x + {}^nC_1D^{n-1}(y_1)D(x)] - m^2D^n(y) = 0$$

$$\Rightarrow \left[(x^2-1)y_{n+2} + 2xny_{n+1} + \frac{n(n-1)(\cancel{2})y_n}{\cancel{2}}\right] + [xy_{n+1} + ny_n] - m^2y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + [n^2 - \cancel{n} + \cancel{n} - m^2]y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + [n^2-m^2]y_n = 0$$

Q8) If $y = e^{\tan^{-1}x}$ then prove that

$$(1+x^2)y_2 + (2x-1)y_1 = 0 \quad \&$$

$$(1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$$

Sol^n $y = e^{\tan^{-1}x}$ diff. w. r. to 'x'

$$\Rightarrow y_1 = e^{\tan^{-1}x}\frac{1}{1+x^2} \quad \Rightarrow y_1 = \frac{y}{1+x^2}$$

$\Rightarrow (1+x^2)y_1 = y$ again diff. w. r. to x

$$\Rightarrow (1+x^2)y_2 + 2xy_1 = y_1$$

$$\Rightarrow (1+x^2)y_2 + 2xy_1 - y_1 = 0$$

$$\Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0$$

diff. n times by Leibnitz's theorem

$$\Rightarrow D^n[(1+x^2)y_2] + D^n[(2x-1)y_1] = 0$$

$$\Rightarrow [D^n(y_2)(1+x^2) + {}^nC_1D^{n-1}(y_2)D(1+x^2) + {}^nC_2D^{n-2}(y_2)D^2(1+x^2)] + [D^n(y_1)(2x-1) + {}^nC_1D^{n-1}(y_1)D(2x-1)] = 0$$

$$\Rightarrow \left[(1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{\cancel{2}}\cancel{2}y_n\right] +$$

$(2x-1)y_{n+1} + 2n y_n] = 0$

$\Rightarrow (1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + [n^2 - n + 2n]y_n$

$\Rightarrow (1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+2} + n(n+1)y_n = 0$

n^{th} Differential Coefficient for Special value at x

Sometime it is difficult to find General value of y_n but its value can be found for special values of x. The values of y_n at $x=0$ is usually denoted by $(y_n)_0$

Q1) If $y = \sin(a \sin^{-1} x)$, then find $(y_n)_0$

Sol: $y = \sin(a\sin^{-1}x)$

$y_1 = \cos(a \sin^{-1}x) \cdot \frac{a}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = a\cos(a\sin^{-1}x)$ [Squaring BS]

$\Rightarrow (1-x^2)y_1^2 = a^2\cos^2(a\sin^{-1}x)$

$\Rightarrow (1-x^2)y_1^2 = a^2[1 - \sin^2(a\sin^{-1}x)]$

$\Rightarrow (1-x^2)y_1^2 = a^2[1 - y^2]$

again diff. w. r. to x.

$\Rightarrow (1-x^2)2y_1y_2 + (-2x)y_1^2 = a^2[-2yy_1]$

$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 + a^2 2yy_1 = 0$

$\Rightarrow 2y_1[(1-x^2)y_2 - xy_1 + a^2y] = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 + a^2y = 0$

diff. n times by Leibnitz's Theorem

$\Rightarrow D^n[(1-x^2)y_2] - D^n(xy_1) + D^n(a^2y) = 0$

$\Rightarrow [D^n(y_2)(1-x^2) + {}^nC_1 D^{n-1}(y_2)D(1-x^2) + {}^nC_2 D^{n-2}(y_2) D^2(1-x^2)] - [D^n(y_1)x + {}^nC_1 D^{n-1}(y_1)D(x)] + [D^n(y)a^2] = 0$

$\Rightarrow \left[(1-x^2)y_{n+2} + (-2x)n y_{n+1} + \frac{n(n-1)}{2}(-2)y_n\right] - [xy_{n+1} + ny_n] + a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2xny_{n+1} + n(n-1)y_n - xy_{n+1} - ny_n + a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - \cancel{n} + \cancel{n} - a^2)y_n$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-a^2)y_n = 0$

$\Rightarrow (y_{n+2})_{x=0} = (n^2-a^2)(y_n)_{x=0}$

$$\left\{\begin{array}{l} \because y_1 = \cos(a\sin^{-1}x)\cdot\frac{a}{\sqrt{1-x^2}} \quad -(1) \\ (1-x^2)y_2 - xy_1 + a^2y = 0 \quad -(2) \end{array}\right\}$$

$\Rightarrow (y_1)_{x=0} = \cos(a\sin^{-1}0)\cdot\frac{a}{\sqrt{1-0^2}}$

$\Rightarrow (y_1)_{x=0} = a \quad \{\because \cos 0 = 1\}$

Putting $x=0$ in (2)

$\Rightarrow (y_2)_{x=0} = 0$

$\Rightarrow (y_{n+2})_{x=0} = (n^2-a^2)(y_n)_{x=0} \quad -(3)$

for odd

1) Put $n=1$ in (3)

$(y_3)_0 = (1-a^2)(y_1)_0$

$(y_3)_0 = a(1-a^2)$

2) Put $n=3$ in (3)

$(y_5)_0 = (9-a^2)(y_3)_0$

$(y_5)_0 = a(1-a^2)(9-a^2)$

for even

1) Put $n=2$ in (3)

2) $(y_4)_0 = (4-a^2)(y_2)_0 = 0$

3) $(y_6)_0 = 0$ 4) $(y_8)_0 = 0$

for odd :- $(Y_n)_0 = a(1-a^2)(3^2-a^2)\ldots((n-1)^2-a^2)$
for even :- $(Y_n)_0 = 0$

Q2) If $Y = \tan^{-1}x$, then prove that
$(1+x^2)Y_{n+2} + 2(n+1)xY_{n+1} + n(n+1)Y_n = 0$
hence find $(Y_n)_0$

Sol $Y = \tan^{-1}x$ — ①

$Y_1 = \frac{1}{1+x^2}$ — ② $\Rightarrow (1+x^2)Y_1 = 1$
again diff. w.r. to x

$\Rightarrow (1+x^2)Y_2 + Y_1 2x = 0$ — ③

diff. n times w.r. to Leibnitz's theo.

$\Rightarrow D^n((1+x^2)Y_2) + D^n(2xY_1) = 0$

$\Rightarrow [D^n(Y_2)(1+x^2) + {}^nC_1 D^{n-1}(Y_2)D(1+x^2) + {}^nC_2 D^{n-2}(Y_2)D^2(1+x^2)] + [D^n(Y_1)2x + {}^nC_1 D^{n-1}(Y_1) D(2x)] = 0$

$\Rightarrow [(1+x^2)Y_{n+2} + 2xnY_{n+1} + \frac{n(n-1)}{2}\times 2\, Y_n + 2xY_{n+1} + n2Y_n] = 0$

$\Rightarrow (1+x^2)Y_{n+2} + 2(n+1)xY_{n+1} + [n^2-n+2n]Y_n = 0$

$\Rightarrow (1+x^2)Y_{n+2} + 2(n+1)xY_{n+1} + n(n+1)Y_n = 0$

Putting $x=0$, then we get

$\Rightarrow (Y_{n+2})_0 = -n(n+1)(Y_n)_0$ — ④

from eqn ①, ② & ③, we get

$\Rightarrow (Y)_0 = \tan^{-1}0 = 0$

$\Rightarrow (Y_1)_0 = \frac{1}{1+0} = 1$, $(Y_2)_0 = 0$.

$\Rightarrow$ Putting $n = 2, 4, 6 \ldots$ in eqn (4) we get

① If $n=2$:- $(Y_4)_0 = -2(2+1)(Y_2)_0$
$\Rightarrow (Y_4)_0 = 0$ $\{\because (Y_2)_0 = 0\}$

② If $n=4$:- $(Y_6)_0 = -4(4+1)(Y_4)_0$
$\Rightarrow (Y_6)_0 = 0$ $\{\because (Y_4)_0 = 0\}$

Same $(Y_8)_0 = (Y_{10})_0 = \ldots = 0$

$\Rightarrow$ Putting $n = 1, 3, 5, 7 \ldots$ in eqn ④ we get

(i) If $n=1$:- $(Y_3)_0 = -1(1+1)(Y_1)_0$
$\Rightarrow (Y_3)_0 = -1.2\times 1 \Rightarrow (Y_3)_0 = (-1)2!$

(ii) If $n=3$:- $(Y_5)_0 = -3.4(Y_3)_0$
$\Rightarrow (Y_5)_0 = -3.4.2!(-1) \Rightarrow (Y_5)_0 = (-1)^2.4!$

$\therefore$ General way when n is even :- $(Y_n)_0 =$

$\therefore$ General way when n is odd :-
$(Y_n)_0 = (-1)^{(n-1)/2}(n-1)!$

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Chapter-02

Maclaurins & Taylor's Theorem

Maclaurins Theorem:- If $f(x)$ be a function of the variable x such that it can be expanded in ascending power at x & the expansion be differentiable any number of times then $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^n(0)$

ex:- ① $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$

② $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$

③ $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} \dots$

Q1) Expand $\log(1+x)$ in powers of x by maclaurin theorem

Sol^n let $Y = \log(1+x)$

$\Rightarrow$ at $Y(0) = \log 1 \Rightarrow Y(0) = 0$

$\Rightarrow Y_1 = \frac{1}{1+x}, \quad Y_1(0) = 1$

$\Rightarrow Y_2 = -\frac{1}{(1+x)^2}, \quad Y_2(0) = -1$

$\Rightarrow Y_3 = \frac{2}{(1+x)^3}, \quad Y_3(0) = 2$

$\left\{ \text{Maclaurin series :-} \quad Y(x) = Y(0) + \frac{x}{1!}Y_1(0) + \frac{x^2}{2!}Y_2(0) + \frac{x^3}{3!}Y_3(0) + \dots + \frac{x^n}{n!}Y_n(0) \dots \right\}$

$\Rightarrow \log(1+x) = \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} \dots$

Q2) Expand $Y = e^x \cos x$ in powers of x by maclaurin theorem

Sol^n $Y = e^x \cos x, \quad Y(0) = 1$

$\Rightarrow Y_1 = -e^x \sin x + e^x \cos x, \quad Y_1(0) = 1$

$\Rightarrow Y_1 = Y - e^x \sin x$

$\Rightarrow Y_2 = Y_1 - e^x \cos x \mp e^x \sin x,$

$\Rightarrow Y_2(0) = 1 - 1 = 0$

$\Rightarrow Y_2 = Y_1 - Y \mp e^x \sin x$

$\Rightarrow Y_3 = Y_2 - Y_1 \mp e^x \cos x \mp e^x \sin x$

$\Rightarrow Y_3 = 0 - 1 - 1 - 0 \Rightarrow Y_3 = -2$

$\left\{ \because Y = Y(0) + \frac{x}{1!}Y_1(0) + \frac{x^2}{2!}Y_2(0) + \frac{x^3}{3!}Y_3(0) + \dots \right\}$

$\Rightarrow e^x \cos x = 1 + \frac{x}{1!} - \frac{2x^3}{3!}$ ans

Q3) Apply Maclaurin theorem & prove that $\log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots$

Sol^n $Y = \log(\sec x), \quad Y(0) = \log 1 = 0 \{\because \sec 0 =$

$\Rightarrow Y_1 = \frac{1}{\sec x} \times \sec x . \tan x \Rightarrow Y_1 = \tan x$

$\Rightarrow Y_1(0) = 0$

$\Rightarrow Y_2 = \sec^2 x, \quad Y_2(0) = 1$

$\Rightarrow Y_3 = 2\sec x (\sec x . \tan x), \quad Y_3(0) = 0$

$\Rightarrow Y_3 = 2\sec^2 x \tan x$

$\Rightarrow y_4 = 2\sec^2 x \sec^2 x + 2\tan x(2\sec x)(\sec x \tan x)$

$\Rightarrow y_4(0) = 2$

$\Rightarrow y_4 = 2\sec^4 x + 4\tan^2 x \sec^2 x$

$\Rightarrow y_5 = 8\sec^3 x . \sec x \tan x + 8\tan x(\sec^2 x)\sec^2 x + 4\tan^2 x \, 2\sec x(\sec x . \tan x)$

$\Rightarrow y_5(0) = 0$

$\Rightarrow y_6 = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x$

$\Rightarrow y_6(0) = 16.$

$\left\{ \because y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots \right\}$

$\Rightarrow \log(\sec x) = \frac{1}{2}x^2 + \frac{2x^4}{4\times3\times2\times1} + \frac{16x^6}{6\times5\times4\times3\times2\times1}$

$\Rightarrow \log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6$ H.P.

ans

Q4) find the first five terms in the expansion of $e^{\sin x}$ by Maclaurin's theorem

Sol: $y = e^{\sin x}$, $\boxed{(y)_0 = 1}$

$y_1 = \cos x . e^{\sin x}$ $\boxed{(y_1)_0 = 1}$

$\Rightarrow y_1 = y\cos x$

$\Rightarrow y_2 = y_1\cos x - y\sin x$, $(y_2)_0 = 1 - 0 = 1$

$\Rightarrow y_3 = y_2\cos x - y_1\sin x - y_1\sin x - y\cos x$

$\Rightarrow y_3 = y_2\cos x - 2y_1\sin x - y\cos x$

$\Rightarrow (y_3)_0 = 1 - 0 - 1 \Rightarrow \boxed{(y_3)_0 = 0}$

$\Rightarrow y_4 = y_3\cos x - y_2\sin x - 2y_2\sin x - 2y_1\cos x - y_1\cos x + y\sin x$

$\Rightarrow y_4 = y_3\cos x - 3y_2\sin x - 3y_1\cos x + y\sin x$

$\Rightarrow (y_4)_0 = 0 - 0 - 3 + 0 \Rightarrow \boxed{(y_4)_0 = -3}$

$\Rightarrow (y_5) = y_4\cos x - y_3\sin x - 3y_3\sin x - 3y_2\cos x - 3y_2\cos x + 3y_1\sin x + y_1\sin x + y\cos x$

$\Rightarrow y_5 = y_4\cos x - 4y_3\sin x - 6y_2\cos x + 4y_1\sin x + y\cos x$

$\Rightarrow (y_5)_0 = -3 - 0 - 6 + 0 + 1 \Rightarrow \boxed{(y_5)_0 = -8}$

BY Maclaurin's Theoram:-

$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots$

$\Rightarrow e^{\sin x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!}$

$\Rightarrow e^{\sin x} = 1 + x + \frac{x}{2} - \frac{3x^4}{4\times3\times2} - \frac{8x^5}{5\times4\times3\times2}$

$\Rightarrow e^{\sin x} = 1 + x + \frac{x}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \ldots$

ans.

Q5) expand $\tan^{-1}x$ in ascending powers of x by Maclaurin's theoram

Sol: $y = \tan^{-1}x$, $\boxed{(y)_0 = 0}$

$y_1 = \frac{1}{1+x^2}$ $\boxed{(y_1)_0 = 1}$

$\Rightarrow (1+x^2)y_1 = 0$

$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0$ —(1), $(y_2)_0 = 0$

Diff. n times by Leibnitz's theoram

$\Rightarrow D^n[(1+x^2)y_2] + D^n[2xy_1] = 0$

$\Rightarrow [D^n(y_2)(1+x^2) + {}^nC_1 D^{n-1}(y_2) D(1+x^2) + {}^nC_2 D^{n-2}(y_2) D(1+x^2)] + [D^n(y_1)2x + {}^nC_1 D^{n-1}(y_1) D(2x)]$

$\Rightarrow \left[(1+x^2)y_{n+2} + n2x y_{n+1} + \frac{n(n-1)}{2} . 2y_n\right] +$

$2xy_{n+1} + 2ny_n] = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + [n^2 - n + 2n]y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + [n(n+1)]y_n = 0$

Putting $x = 0$ then we get

$\Rightarrow (y_{n+2})_0 = -n(n+1)(y_n)_0$ — ②

Putting $n = 1, 2, 3, 4 \ldots$ in eqⁿ ②

$\Rightarrow (y_3)_0 = -1(1+1)(y_1)_0 \quad \{\because (y_1)_0 = 1\}$

$\Rightarrow (y_3)_0 = (-1) \cdot 2 \cdot 1 \Rightarrow (y_3)_0 = -2!$

$\Rightarrow (y_4)_0 = -2(2+1)(y_2)_0 \quad \{\because (y_2)_0 = 0\}$

$\Rightarrow (y_4)_0 = 0$

$\Rightarrow (y_5)_0 = -3(3+1)(y_3)_0 \quad \because (y_3)_0 = -2!$

$\Rightarrow (y_5)_0 = -3 \cdot (4) \cdot (-2!) = 3 \cdot 4 \cdot 2 = 4!$

$\Rightarrow (y_6)_0 = -4(4+1)(y_4)_0 \quad \{\because (y_4)_0 = 0\}$

$\Rightarrow (y_6)_0 = 0$

$$\left\{\because y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots\right\}$$

$$\Rightarrow \tan^{-1}x = x - \frac{\cancel{2}x^3}{3\times\cancel{2}} + \frac{\cancel{4!}\,x^5}{5\times\cancel{4!}}$$

$$\Rightarrow \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5}$$

Q6) Expand $e^{a\sin^{-1}x}$ by Maclaurin's Theorem show that $e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2\sin^3\theta}{3!}$

Solⁿ $y = e^{a\sin^{-1}x}$ — ①, $(y)_0 = 1$

$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$ $\quad \boxed{(y_1)_0 = a}$

$\Rightarrow (\sqrt{1-x^2})y_1 = ae^{a\sin^{-1}x}$

$\Rightarrow (\sqrt{1-x^2})y_1 = ay$ (applying squaring both side)

$\Rightarrow (1-x^2)y_1^2 = a^2y^2$ ② [again diff. w.r. to x]

$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 = a^2 2yy_1$

$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$ — ③

Put $x = 0$

$\Rightarrow (y_2)_0 = a^2(y)_0 \Rightarrow \boxed{(y_2)_0 = a^2} \quad \{\because (y)_0 = 1\}$

again diff. eqⁿ ③ w.r. to x

$\Rightarrow (1-x^2)y_3 - 2xy_2 - y_1 - xy_2 - a^2y_1 = 0$

Put $x = 0$

$\Rightarrow (y_3)_0 = a^2(y_1)_0 + (y_1)_0 \Rightarrow (y_3)_0 = a^3 + a$

$\boxed{(y_3)_0 = a(1+a^2)}$

Similarly $\boxed{(y_4)_0 = a^2(2^2 + a^2)}$

$$\left\{\because y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots\right\}$$

$$\Rightarrow e^{a\sin^{-1}x} = 1 + \frac{ax}{1!} + \frac{a^2x^2}{2!} + \frac{a(1+a^2)x^3}{3!} + \frac{a^2(2^2+a^2)}{4!}$$

$\Rightarrow$ Put $a = 1$ & $\sin^{-1}x = \theta$ or $x = \sin\theta$

$$\Rightarrow e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2\sin^3\theta}{3!} + \frac{5}{4!}\sin^4\theta$$ ans

Q7) Prove $\log\cosh x = \frac{x^2}{2!} - \frac{x^4}{12} + \frac{x^6}{45}$

Solⁿ $y = \log\cosh x \Rightarrow \boxed{(y)_0 = 0}$

$\Rightarrow y_1 = \frac{1}{\cosh x} \cdot \sinh x \Rightarrow y_1 = \tanh x \Rightarrow \boxed{(y_1)_0 = 0}$

$\Rightarrow y_2 = \operatorname{sech}^2 x \Rightarrow \boxed{(y_2)_0 = 1} \quad \{\because \operatorname{sech}^2 x = 1 - \tanh^2 x$

$\Rightarrow y_2 = 1 - \tanh^2 x \Rightarrow y_2 = 1 - y_1^2$

$\Rightarrow y_3 = -2y_1y_2 \Rightarrow (y_3)_0 = -2\times0\times1 \Rightarrow \boxed{(y_3)_0 = 0}$

$\Rightarrow y_4 = -2y_1y_3 - 2y_2y_2 \Rightarrow (y_4)_0 = -2\times0\times0 - 2\times1\times1$

$\Rightarrow \boxed{(Y_4)_0 = -2}$

$\Rightarrow Y_5 = -2Y_1Y_4 - 2Y_2Y_3 - 4Y_2Y_3$

$\Rightarrow Y_5 = -6Y_2Y_3 - 2Y_1Y_4 \Rightarrow (Y_5)_0 = 6\times1\times0 - 2\times0\times{-2}$

$\Rightarrow \boxed{(Y_5)_0 = 0}$

$\Rightarrow Y_6 = -6Y_2Y_4 - 6Y_3^2 - 2Y_1Y_5 - 2Y_2Y_4$

$\Rightarrow Y_6 = -8Y_2Y_4 - 2Y_1Y_5 - 6Y_3^2$

$\Rightarrow Y_6 = -2(4Y_2Y_4 + Y_1Y_5 + 3Y_3^2)$

$\Rightarrow (Y_6)_0 = -2(4\times1\times(-2) + 0\times0 + 0)$

$\Rightarrow (Y_6)_0 = -2\times(-8) \Rightarrow (Y_6)_0 = 16$

$\because Y = (Y)_0 + \frac{x}{1!}(Y_1)_0 + \frac{x^2}{2!}(Y_2)_0 + \frac{x^3}{3!}(Y_3)_0 + \dots$

$\Rightarrow \log\cosh x = 0 + 0 + \frac{x^2}{2!} + 0 - \frac{2x^4}{4\times3\times2\times1} + 0 + \frac{16x^6}{6\times5\times4\times3\times2}$

$\Rightarrow \log\cosh x = \frac{x^2}{2!} - \frac{x^4}{12} + \frac{x^6}{45}$ ans.

~~Q8) Expand log sinx in powers of (x-2)~~

Taylor's Theorem:- If $(a+h)$ be a function of the variable h such that it can be expanded in ascending powers of h and this expansion be differentiable any number of times then

$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \dots$

$\Rightarrow$ let $(a+h) = x \Rightarrow h = x-a$

$\Rightarrow f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a)$

Put $a=0$ in Taylor's series formula

$\Rightarrow f(h) = f(0) + \frac{h}{1!}f'(0) + \frac{h^2}{2!}f''(0) + \dots$

This is maclaurin's series

Q1) Expand log sinx in power of $(x-2)$

Sol^n $a = 2$, $h = x-2$

$\Rightarrow f(2+x-2) = f(2) + \frac{(x-2)}{1!}f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \dots$

$\Rightarrow \{\because f(x) = \log\sin x\} \Rightarrow f'(x) = \cot x$

$\Rightarrow f''(x) = -\operatorname{cosec}^2 x \Rightarrow f'''(x) = -2\operatorname{cosec} x.(-\operatorname{cosec} x.\cot x)$

$\Rightarrow f'''(x) = 2\operatorname{cosec}^2 x \cot x$

$\Rightarrow f(2) = \log\sin 2 \Rightarrow f'(2) = \cot 2$

$\Rightarrow f''(2) - \operatorname{cosec}^2 2 \Rightarrow f'''(2) = 2\operatorname{cosec}^2 2 \cot 2$

$\Rightarrow f(x) = \log\sin 2 + \frac{(x-2)}{1!}\cot 2 + \frac{(x-2)^2}{2!}(-\operatorname{cosec}^2 2) + \frac{(x-2)^3}{3!}2\operatorname{cosec}^2 2\cot 2$

ans.

Q2) Expand $\tan^{-1}x$ in power of $(x-\pi/4)$ by Taylor's Theorem

Sol^n $a = \pi/4$, $h = x-\pi/4$

$\Rightarrow f(\pi/4 + x - \pi/4) = f(\pi/4) + \frac{(x-\pi/4)}{1!}f'(\pi/4) + \frac{(x-\pi/4)^2}{2!}f''(\pi/4) + \frac{(x-\pi/4)^3}{3!}f'''(\pi/4)$

$\Rightarrow \because f(x) = \tan^{-1}x \Rightarrow f(\pi/4) = 1$

$\Rightarrow f'(x) = \frac{1}{1+x^2} \quad \Rightarrow f'(\pi/4) = \frac{1}{1+\pi^2/16} = \frac{16}{\pi^2+16}$

$\Rightarrow f''(x) = \frac{-2x}{(1+x^2)^2} \quad \Rightarrow f''(\pi/4) = \frac{-2\pi/4}{(1+\pi^2/16)^2}$

$\Rightarrow f''(\pi/4) = \frac{-\pi}{2}\frac{(16)^2}{(16+\pi^2)^2} \quad \Rightarrow f''(\pi/4) = \frac{-128\pi}{(16+\pi^2)^2}$

$\Rightarrow f(x) = 1 + \frac{(x-\pi/4)}{1!}\frac{16}{\pi^2+16} + \frac{(x-\pi/4)^2}{2!}\left[\frac{-128}{(16+\pi^2)^2}\right] + \dots$

$\Rightarrow f(x) = 1 + \frac{(x-\pi/4)}{1!}\frac{16}{\pi^2+16} - \frac{(x-\pi/4)^2}{2!}\frac{128}{(16+\pi^2)^2}$

Q3) By Taylor's Theoram, Prove that
$\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3}$

Sol^n $a = h$, $h = x$

$\Rightarrow f(x+h) = f(h) + \frac{(x)}{1!}f'(h) + \frac{(x)^2}{2!}f''(h)$

$\because f(h) = \log h \quad \Rightarrow f'(h) = 1/h$

$\Rightarrow f''(h) = \frac{-1}{h^2} \quad \Rightarrow f'''(h) = \frac{2}{h^3}$

$\Rightarrow \log(x+h) = \log h + \frac{x}{1!}\frac{1}{h} - \frac{x^2}{2!}\frac{1}{h^2} + \frac{x^3}{3\cancel{\times 2}}\frac{\cancel{2}}{h^3}$

$\Rightarrow \log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3}$

Q4) Expand $2x^3+7x^2+x-1$ in powers of $(x-2)$ by Taylor's theoram

Sol^n $a = 2$, $h = x-2$

$f(2+x-2) = f(2) + \frac{(x-2)}{1!}f'(2) + \frac{(x-2)^2}{2!}f''(2) + \dots$

$\because f(x) = 2x^3 + 7x^2 + x - 1$

$\Rightarrow f(2) = 2\times 8 + 7\times 4 + 2 - 1 \quad \Rightarrow f(2) = 16 + 28 + 1$

$\Rightarrow f(2) = 45$

$\Rightarrow f'(x) = 6x^2 + 14x + 1 \quad \Rightarrow f'(2) = 6\times 4 + 14\times 2 + 1$

$\Rightarrow f'(2) = 24 + 28 + 1 \quad \Rightarrow f'(2) = 53$

$\Rightarrow f''(x) = 12x + 14 \quad \Rightarrow f''(2) = 12\times 2 + 14$

$\Rightarrow f''(2) = 38$

$\Rightarrow f'''(x) = 12 \quad \Rightarrow f'''(2) = 12$

$\Rightarrow f(x) = 45 + \frac{(x-2)}{1!}53 + \frac{(x-2)^2}{2\times 1}\cancel{38}^{19} + \frac{(x-2)^3}{3\times 2\times 1}\cancel{12}^{2}$

$\Rightarrow f(x) = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$ ans

Q5) Expand $\sin x$ in power of $(x - \pi/2)$

Sol^n $a = \pi/2$, $h = x - \pi/2$

$\Rightarrow f(\pi/2 + x - \pi/2) = f(\pi/2) + \frac{(x-\pi/2)}{1!}f'(\pi/2) + \frac{(x-\pi/2)^2}{2!}f''(\pi/2) + \frac{(x-\pi/2)^3}{3!}f'''$

$\because f(x) = \sin x \quad \Rightarrow f(\pi/2) = \sin\pi/2 = \cancel{1/\sqrt{2}}\ 1$

$\Rightarrow f'(x) = \cos x \quad \Rightarrow f'(\pi/2) = \cos\pi/2 = \cancel{1/\sqrt{2}}\ 0$

$\Rightarrow f''(x) = -\sin x \quad \Rightarrow f''(\pi/2) = -\sin\pi/2 = \cancel{-1/\sqrt{2}}\ -1$

$\Rightarrow f'''(x) = -\cos x \quad \Rightarrow f'''(\pi/2) = -\cos\pi/2 = -1/\sqrt{2}$

$\Rightarrow f(x) = \frac{1}{\sqrt{2}} + \frac{(x-\pi/2)}{1!} + \frac{(x-\pi/2)^2}{2!}\frac{1}{\sqrt{2}} - \frac{(x-\pi/2)^3}{3!}\frac{1}{\sqrt{2}}$

$$= -\frac{(x-\pi/2)^4}{4!} \cdot \frac{1}{\sqrt{2}}$$ wrong

Sol^n $a = \pi/2$, $h = x - \pi/2$

$$f(\pi/2 + x - \pi/2) = f\left[\frac{\pi}{2}\right] + \left(x - \frac{\pi}{2}\right) f'\frac{\pi}{2}$$

$$+ \frac{(x-\pi/2)^2}{2!} f''(\pi/2) + \ldots$$

$\Rightarrow f(x) = \sin x \quad \Rightarrow f(\pi/2) = \sin\pi/2 = 1$

$\Rightarrow f'(x) = \cos x \quad \Rightarrow f'(\pi/2) = \cos\pi/2 = 0$

$\Rightarrow f''(x) = -\sin x \quad \Rightarrow f''(\pi/2) = -\sin\pi/2 = -1$

$\Rightarrow f'''(x) = -\cos x \quad \Rightarrow f'''(\pi/2) = -\cos\pi/2 = 0$

$\Rightarrow f''''(x) = \sin x \quad \Rightarrow f''''(\pi/2) = \sin\pi/2 = 1$

$$\Rightarrow \sin x = 1 + (x-\pi/2)\cdot 0 + \frac{(x-\pi/2)^2}{2!}\cdot(-1) + \frac{(x-\pi/2)^3}{3!}\cdot 0 + \frac{(x-\pi/2)^4}{4!}\cdot 1$$

$$\Rightarrow \sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!}$$

Chapter-03

Indeterminate form

There are total 7 Indeterminate form i.e $0/0$, ∞/∞, $0\times\infty$, $\infty-\infty$, 1^∞, 0^0 & ∞^0

$\Rightarrow$ L Hospital's Rule :- This rule says, diff N & D separately.

formula:- $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0} \frac{f'(x)}{g'(x)}$

Q1) $\lim\limits_{x\to 0} \frac{\sin x}{x}$

Sol^n putting $x=0$ in $\frac{\sin x}{x}$ then we get $0/0$ form

applying L Hospital's rule

$\lim\limits_{x\to 0} \frac{\cos x}{1} \Rightarrow \frac{\cos 0}{1} \Rightarrow \frac{1}{1} \Rightarrow 1$

Q2) $\lim\limits_{x\to 0} \frac{x - \sin x}{x^3}$

Sol^n This is a $[0/0]$ form

applying L Hospital's rule

$\Rightarrow \lim\limits_{x\to 0} \frac{1-\cos x}{3x^2}$ {$[0/0]$ form}

again applying L Hospital's rule

$\Rightarrow \lim\limits_{x\to 0} \frac{0+\sin x}{6x}$ $0/0$ form

again applying L H rule

$\lim_{x\to 0} \frac{\cos x}{6x} \Rightarrow \frac{\cos 0}{6} \Rightarrow \frac{1}{6}$ // ans

Q3) $\lim_{x\to\infty} \frac{x^2+3x+4}{3x^2+2x+5}$

Sol: It is an ∞/∞ form
applying LH rule

$\lim_{x\to\infty} \frac{2x+3}{6x+2}$ again by LH rule

$\Rightarrow \lim_{x\to\infty} \frac{2}{6} \Rightarrow \frac{\cancel{2}}{3\cancel{6}}$ // ans or $\frac{1}{3}$

Q4) $\lim_{x\to 0} \frac{x-\sin x}{\tan^3 x}$

Soln this is 0/0 form
applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{1-\cos x}{3\tan^2 x \sec^2 x}$

$\Rightarrow \lim_{x\to 0} \frac{0+\sin x}{6\tan x \cdot \sec^2 x}$

$\Rightarrow \lim_{x\to 0} \frac{x-\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right\}}{\left\{x+\frac{x^3}{3}+\frac{2}{15}x^5+\dots\right\}^3}$

$\Rightarrow \lim_{x\to 0} \frac{x^3\left\{\frac{1}{6}-\frac{x^5}{5!}+\dots\right\}}{x^3\left\{1+\frac{x^2}{3}+\frac{2x^5}{15}+\dots\right\}^3} = \frac{1}{6}$ ans

Ques) $\Rightarrow [0\times\infty]$ this form is firstly converted into $\frac{0}{1/\infty}$ and then into $\frac{0}{0}$ form or in $\frac{\infty}{1/0}$ and then $\frac{\infty}{\infty}$ form

so basically $0\times\infty$ form has to converted into 0/0 form or ∞/∞

Q1) $\lim_{x\to 0} x\cdot\log x$

Soln this is $0\times\infty$ form

$\lim_{x\to 0} \frac{\log x}{1/x}$

applying LH rule

$\lim_{x\to 0} \frac{\frac{1}{x}}{-1/x^2} \Rightarrow \lim_{x\to 0} -x$

$\Rightarrow$ 0 ans

$\Rightarrow \infty-\infty$ form $= \lim_{x\to a} [f(x)-g(x)]$

$\Rightarrow \lim_{x\to a} \left\{\frac{\frac{1}{f(x)}-\frac{1}{g(x)}}{\frac{1}{f(x)\,g(x)}}\right\}$ after this $\infty-\infty$ will change into either 0/0 or ∞/∞

Q1) $\lim_{x\to\pi/2} [\sec x-\tan x]$

Sol^n $\lim_{x\to\pi/2}\left[\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right]$

$\Rightarrow \lim_{x\to\pi/2}\left[\frac{1-\sin x}{\cos x}\right]$ Now this is 0/0 form

$\Rightarrow$ applying LH rule

$\Rightarrow \lim_{x\to\pi/2}\frac{0+\cos x}{+\sin x} \Rightarrow \frac{\cos\pi/2}{\sin\pi/2} \Rightarrow \frac{0}{1}$

ans = 0

$\Rightarrow$ Methods for solving 0/0 or ∞/∞ form

- factorisation method
- Rationalisation method
- L'Hopital rule

Q1) $\lim_{x\to1} 3x^2+4x+5$

Sol^n $3(1)^2+4(1)+5 \Rightarrow 3+4+5 \Rightarrow 12$ // ans

Q2) evaluate $\lim_{x\to2}\frac{x^2-5x+6}{x^2-4}$

Sol^n this is 0/0 form

(I) (doing this by factorisation)

$\Rightarrow \lim_{x\to2}\frac{x^2-2x-3x+6}{x^2-4} \Rightarrow \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)}$

$\Rightarrow \lim_{x\to2}\frac{x-3}{x+2} \Rightarrow \frac{2-3}{2+2} \Rightarrow -\frac{1}{4}$ ans

(2) doing this by L'Hopital rule

$\Rightarrow \lim_{x\to2}\frac{x^2-5x+6}{x^2-4}$ applying L'Hopital rule

$\Rightarrow \lim_{x\to2}\frac{2x-5}{2x} \Rightarrow \frac{2\times2-5}{2\times2} \Rightarrow \frac{-1}{4}$ ans

Q3) evaluate $\lim_{x\to1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)}$

Sol^n this is 0/0 form

$\lim_{x\to1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)}$

$\Rightarrow$ applying L'Hopital rule

$\Rightarrow \lim_{x\to1}\frac{2x+x\times1/x+\log x-1/x}{2x}$

$\Rightarrow \lim_{x\to1}\frac{2x+1+\log_e x-1/x}{2x}$

$\Rightarrow \frac{2\times1+\cancel{1}+\log1-\cancel{1/1}}{2\times1} \Rightarrow \frac{2}{2} \Rightarrow 1$ //

Q4) evaluate $\lim_{x\to\pi/4}\frac{1-\sin 2x}{1+\cos 4x}$

this is 0/0 form

$\{\because \sin^2x+\cos^2x=1$ & $\sin 2x = 2\sin x\cos x\}$

$\Rightarrow \lim_{x\to\pi/4} \dfrac{\sin^2x+\cos^2x-2\sin x\cos x}{2(\cos x-\sin x)^2(\cos x+\sin x)^2}$

$\because 1+\cos 4x = 2\cos^2 2x = 2(\cos^2x-\sin^2x)^2$

$1+\cos 4x = 2(\cos x-\sin x)^2(\cos x+\sin x)^2$

$\Rightarrow \lim_{x\to\pi/4} \dfrac{\cancel{(\cos x-\sin x)^2}}{2\cancel{(\cos x-\sin x)^2}(\cos x+\sin x)^2}$

$\Rightarrow \dfrac{1}{2(\cos\pi/4+\sin\pi/4)^2} \Rightarrow \dfrac{1}{2(1/\sqrt{2}+1/\sqrt{2})^2}$

$\Rightarrow \dfrac{1}{2\left[\frac{2}{\sqrt{2}}\right]^2} \Rightarrow \dfrac{1}{2(\sqrt{2})^2} \Rightarrow \dfrac{1}{4}$ ans//

$\Rightarrow$ Rationalisation method

Q5) evaluate $\lim_{x\to0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

Sol^n rationalising numerator

$\Rightarrow \lim_{x\to0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x} \times \dfrac{(\sqrt{2+x}+\sqrt{2})}{(\sqrt{2+x}+\sqrt{2})}$

$\Rightarrow \lim_{x\to0} \dfrac{\cancel{2}+x-\cancel{2}}{x(\sqrt{2+x}+\sqrt{2})} \Rightarrow \lim_{x\to0}\dfrac{\cancel{x}}{\cancel{x}(\sqrt{2+x}+\sqrt{2})}$

$\Rightarrow \lim_{x\to0} \dfrac{1}{\sqrt{2+x}+\sqrt{2}} \Rightarrow \dfrac{1}{2\sqrt{2}}$ ans//

again doing this ques with L'Hopital rule

$\Rightarrow \lim_{x\to0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

$\Rightarrow \lim_{x\to0} \dfrac{\frac{1}{2\sqrt{2+x}}-0}{1} \Rightarrow \dfrac{1}{2\sqrt{2}}$ ans//

Q6) evaluate $\lim_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

Sol^n this is 0/0 form

Rationalising N & D

$\Rightarrow \lim_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \dfrac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}$

$\Rightarrow \lim_{x\to a} \dfrac{(a+2x-3x)\,\sqrt{3a+x}+2\sqrt{x}}{(3a+x-4x)\,\sqrt{a+2x}+\sqrt{3x}}$

$\Rightarrow \lim_{x\to a} \dfrac{\cancel{(a-x)}\,\sqrt{3a+x}+2\sqrt{x}}{3\cancel{(a-x)}\,\sqrt{a+2x}+\sqrt{3x}}$

$\Rightarrow \dfrac{1}{3}\dfrac{\sqrt{3a+a}+2\sqrt{a}}{\sqrt{a+2a}+\sqrt{3a}} \Rightarrow \dfrac{1}{3}\dfrac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$\Rightarrow \dfrac{2\cancel{4}\cancel{\sqrt{a}}}{3\times\cancel{2}\sqrt{3}\cancel{\sqrt{a}}} \Rightarrow \dfrac{2}{3\sqrt{3}}$ ans//

$\Rightarrow$ If $\lim_{x\to\infty}$ this is giving then put $x=\dfrac{1}{t}$

then $\Rightarrow t=\dfrac{1}{x}$ if $x\to\infty$ then $t\to0$

If $\lim_{x\to-\infty}$, put $x=-\dfrac{1}{t}$ then $t\to0$

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Q1) $\lim_{x\to\infty} \frac{2x+3}{5x-4}$

Sol: put $x = 1/t$

$\lim_{t\to 0} \frac{\frac{2}{t}+3}{\frac{5}{t}-4} \Rightarrow \lim_{t\to 0} \frac{2+3t}{5-4t}$

$\Rightarrow 2/5$ ans//

Second method to solve this

$\Rightarrow$ divide by the highest power of x

$\Rightarrow \lim_{x\to\infty} \frac{2+\frac{3}{x}\to 0}{5-\frac{4}{x}\to 0} \Rightarrow \frac{2}{5}$ ans//

Q2) $\lim_{x\to\infty} \frac{2x^2-x+1}{3x^2+5x-6}$

Sol: $\lim_{x\to\infty} \frac{2-\frac{1}{x}\to 0+\frac{1}{x^2}\to 0}{3+\frac{5}{x}\to 0-\frac{6}{x^2}\to 0}$ [dividing by x^2 in N & D]

$\Rightarrow 2/3$ ans

Short tricks for this type of ques

$$\boxed{\lim_{x\to\infty} \frac{\text{Polynomial}}{\text{Polynomial}}}$$

- If degree of N > degree of D: In this case the ans will always be 'Does not exist' i.e. ∞
- If deg N = deg D: In this case the highest power's coefficient will be the ans
- If deg N < deg D: In this case ans is always be 0

Q3) $\lim_{n\to\infty} \left\{\frac{1}{1-n^2}+\frac{2}{1-n^2}+\dots+\frac{n}{1-n^2}\right\}$ is equals to

Soln $\lim_{n\to\infty} \frac{1+2+\dots+n}{1-n^2} \Rightarrow \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{1-n^2}$

$\Rightarrow \lim_{n\to\infty} \frac{\frac{n^2}{2}+\frac{n}{2}}{-n^2+1}$ dividing by n^2 in N&D

$\Rightarrow \lim_{n\to\infty} \frac{\frac{1}{2}+\frac{1}{2n}\to 0}{-1+\frac{1}{n^2}\to 0} \Rightarrow \frac{-1}{2}$ ans//

Q4) evaluate $\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

Soln $\lim_{x\to\infty} \frac{\cancel{x}\left(\sqrt{3-\frac{1}{x^2}\to 0}-\sqrt{2-\frac{1}{x^2}\to 0}\right)}{\cancel{x}\left(4+\frac{3}{x}\right)}$

$\Rightarrow \dfrac{\sqrt{3}-\sqrt{2}}{4}$ ans//

Q5) $\lim_{x\to\infty}\left[\dfrac{x^2+x+1}{x+1}-ax-b\right]=4$ the find the value of a & b

Sol: $\lim_{x\to\infty} \dfrac{x^2+x+1-(ax+b)(x+1)}{x+1}$

$\Rightarrow \lim_{x\to\infty} \dfrac{x^2+x+1-(ax^2+ax+bx+b)}{x+1}$

$\Rightarrow \lim_{x\to\infty} \dfrac{(1-a)x^2+(1-a-b)x+(1-b)}{x+1}$

[as the ques. says limit of this ques exist it means the degree of numerator is same as degree of ~~numerator~~ denominator it mean $(1-a=0)$] $\Rightarrow a=1$ (1)

$\Rightarrow \lim_{x\to\infty} \dfrac{(1-a-b)x+(1-b)}{x+1}$

coefficient of highest power of x is:-

$\Rightarrow \dfrac{1-a-b}{1}=4$ Putting $a=1$

$\Rightarrow \boxed{b=-4}$

$\Rightarrow$ Question on $1^\infty, 0^0, \infty^0$

Q1) $\lim_{x\to 0} [1+x]^{1/x}$

let $y=\lim_{x\to 0}[1+x]^{1/x}$

taking log both side

$\Rightarrow \log y = \lim_{x\to 0} \log[1+x]^{1/x}$

$\Rightarrow \log y = \lim_{x\to 0} \dfrac{1}{x}\log[1+x]$

$\Rightarrow \log y = \lim_{x\to 0} \dfrac{\log[1+x]}{x}$ [Now, this is 0/0 form]

applying L'Hopital rule

$\Rightarrow \log y = \lim_{x\to 0} \dfrac{1}{1+x}$ $\Rightarrow \log y = 1$

$\Rightarrow y = e^1$ $\{\because \log y = x$ then $y=e^x\}$

$\Rightarrow y = e$ ans//

Q2) $\lim_{x\to 0} (\sin x)^{\tan x}$ [This is 0^0 form]

Sol: let $y=\lim_{x\to 0}(\sin x)^{\tan x}$

taking log both side

$\Rightarrow \log y = \lim_{x\to 0} \log(\sin x)^{\tan x}$

$\Rightarrow \log y = \lim_{x\to 0} \tan x \log(\sin x)$ $\left\{\because \tan x = \dfrac{1}{\cot x}\right\}$

$\Rightarrow \log y = \lim_{x\to 0} \dfrac{\log(\sin x)}{\cot x}$

applying LH rule

$\Rightarrow \log y = \lim_{x\to 0} \dfrac{\frac{1}{\sin x}.\cos x}{-\operatorname{cosec}^2 x}$

$\Rightarrow \log y = \lim_{x\to 0} \dfrac{\frac{\cos x}{\sin x}}{\frac{-1}{\sin^2 x}} \Rightarrow \log y = \lim_{x\to 0} -\sin x \cos x$

$\Rightarrow \log y = 0 \times 1 \quad \Rightarrow y = e^0$

$\Rightarrow y = 1$ ans//

Q3) $\lim_{x\to\infty} \dfrac{\log x}{a^x}, a>1$ (This is ∞/∞ form)

Sol^n applying LH rule,

$\Rightarrow \lim_{x\to\infty} \dfrac{\frac{1}{x}}{a^x \log a} \Rightarrow \lim_{x\to\infty} \dfrac{1}{x\, a^x \log a}$

$\Rightarrow \dfrac{1}{\infty\, a^\infty \log a} \Rightarrow \dfrac{1}{\infty} \Rightarrow 0$ ans

Q4) $\lim_{x\to\infty} \dfrac{x(\log x)^3}{1+x+x^2}$

Sol^n $\lim_{x\to\infty} \dfrac{x(\log x)^3}{1+x+x^2}$ {this is ∞/∞ form} applying LH rule

$\Rightarrow \lim_{x\to\infty} \dfrac{(\log x)^3 + x\,3(\log x)^2 \cdot \frac{1}{x}}{1+2x}$

again applying LH rule

$\Rightarrow \lim_{x\to\infty} \dfrac{3(\log x)^2 \frac{1}{x} + 6(\log x)\frac{1}{x}}{2}$

again applying LH rule

$\Rightarrow \lim_{x\to\infty} \dfrac{6\log x \cdot \frac{1}{x} + \frac{6}{x}}{2} \Rightarrow \lim_{x\to\infty} \dfrac{6\log x + 6}{2x}$

again applying LH rule

$\Rightarrow \lim_{x\to\infty} \dfrac{\frac{6}{x}}{2} \Rightarrow \lim_{x\to\infty} \dfrac{\cancel{6}3}{\cancel{2}x} \Rightarrow \dfrac{3}{\infty} \Rightarrow 0$//

Q5) $\lim_{x\to 1} \left[\dfrac{2}{x^2-1} - \dfrac{1}{x-1}\right]$ [This is $\infty - \infty$ form]

Sol^n $\lim_{x\to 1} \left[\dfrac{2-(x+1)}{(x^2-1)}\right] \Rightarrow \lim_{x\to 1} \dfrac{1-x}{x^2-1}$

$\Rightarrow \lim_{x\to 1} \dfrac{-\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} \Rightarrow \lim_{x\to 1} \dfrac{-1}{x+1} \Rightarrow \dfrac{-1}{2}$

Q6) $\lim_{x\to\pi/2} [\sec x - \tan x]$

Sol^n $\lim_{x\to\pi/2} \left[\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}\right] \Rightarrow \lim_{x\to\pi/2} \left[\dfrac{1-\sin x}{\cos x}\right]$

(Now this is 0/0 form)

$\Rightarrow$ applying LH rule

$\Rightarrow \lim_{x\to\pi/2} \dfrac{+\cos x}{+\sin x} \Rightarrow \dfrac{0}{1} \Rightarrow 0$// ans

$\Rightarrow \infty - \infty$ form

Q1) $\lim_{x\to 1} \left[\frac{1}{\log x} - \frac{x}{\log x}\right]$ [This is $\infty - \infty$ form]

Sol^n $\lim_{x\to 1} \left[\frac{1-x}{\log x}\right]$ / Now this is 0/0 form)

applying LH rule

$\Rightarrow \lim_{x\to 1} \frac{-1}{\frac{1}{x}} \Rightarrow \lim_{x\to 1} (-x) \Rightarrow -1$ //

Q2) $\lim_{x\to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x}\right]$ {This is $\infty - \infty$ form}

Sol^n $\lim_{x\to 0} \frac{x-(e^x-1)}{x(e^x-1)} \Rightarrow \lim_{x\to 0} \frac{x-e^x+1}{xe^x - x}$

Now this is 0/0 form

applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{1-e^x+0}{e^x+xe^x-1}$ again applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{-e^x}{e^x+e^x+xe^x-0} \Rightarrow \frac{-1}{1+1+0} \Rightarrow \frac{-1}{2}$ ans

Q3) $\lim_{x\to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$ [This is $\infty - \infty$ form]

Sol^n $\lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x}$ Now this is 0/0 form

~~applying~~ LH rule :- using expansion

$\Rightarrow \lim_{x\to 0} \frac{\left[\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right)^2 - x^2\right]}{x^2\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right]^2}$ $\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\Rightarrow \lim_{x\to 0} \frac{\cancel{x^2} - 2x \times \frac{x^3}{3!} + \text{other higher powers of } x \; \cancel{-x^2}}{x^2 (x^2 - 2x^4/3! + \text{other})}$

$\Rightarrow \lim_{x\to 0} \frac{\cancel{x^4}(-2/3! + \dots)}{\cancel{x^4}(1 - 2/3!\, x^2 \dots)}$

$\Rightarrow \lim_{x\to 0} \frac{-2/3! + \dots}{1 - 2/3\, x^2 + \dots} \Rightarrow \frac{-2/3!}{1}$

$\Rightarrow \frac{-2}{6} \Rightarrow \frac{-1}{3}$ ans //

$\Rightarrow 0 \times \infty$ form :-

Q1) evaluate $\lim_{x\to 0} x \log \sin x$

Sol^n $\lim_{x\to 0} \frac{\log \sin x}{1/x}$ Now this is ∞/∞ form

$\Rightarrow$ applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{\frac{1}{\sin x} \cos x}{-1/x^2} \Rightarrow \lim_{x\to 0} -\frac{x^2 \cos x}{\sin x}$

again by LH rule

$\Rightarrow \lim_{x\to 0} \frac{-2x\cos x + x^2 \sin x}{\cos x} \Rightarrow \frac{0+0}{1} \Rightarrow 0$ //

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Q3) evaluate $\lim_{x\to 0} x^m (\log x)^n$, $m, n > 0$

Sol^n $\lim_{x\to 0} \frac{(\log x)^n}{1/x^m}$ Now this is ∞/∞ form

applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{n(\log x)^{n-1} \frac{1}{x}}{-m . x^{-m-1}}$

$\Rightarrow \lim_{x\to 0} \frac{-n}{m} \frac{(\log x)^{n-1}}{x^{-m-1+1}} \Rightarrow \lim_{x\to 0} (-1)\frac{n}{m} \frac{(\log x)^{n-1}}{x^{-m}}$

again applying LH rule

$\Rightarrow \lim_{x\to 0} (-1)\frac{n}{m} \frac{(n(-1)(\log x)^{n-2} (1/x)}{-m x^{-m-1}}$

$\Rightarrow \lim_{x\to 0} (-1)^2 \frac{n(n-1)}{m^2} \frac{(\log x)^{n-2}}{x^{-m}}$

using this process upto n times

$\Rightarrow \lim_{x\to 0} \frac{(-1)^n n(n-1)(n-2)\cdots 1}{m^n} \frac{(\log x)^{n-n}}{x^{-m}}$

$\Rightarrow \lim_{x\to 0} \frac{(-1)^n n!}{m^n} \frac{1}{x^{-m}} \Rightarrow \lim_{x\to 0} \frac{(-1)^n n! x^m}{m^n}$

$\Rightarrow \frac{m!}{m^n} \times 0 \Rightarrow 0//$

$\Rightarrow 1^\infty, 0^0, \infty^0$ forms

Q1) evaluate $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

let $y = \lim_{x\to 0} ((\cos x)^{\cot^2 x}$

taking log both side

$\Rightarrow \log y = \lim_{x\to 0} \log ((\cos x)^{\cot^2 x}$

$\Rightarrow \log y = \lim_{x\to 0} \cot^2 x \, \log ((\cos x)$

$\Rightarrow \log y = \lim_{x\to 0} \frac{\log \cos x}{\tan^2 x}$ $\left\{ \because \tan x = \frac{1}{\cot x} \right\}$

Now this is $0/0$ form

applying LH rule

$\Rightarrow \log y = \lim_{x\to 0} \frac{-\frac{\sin x}{\cos x}}{2\tan x . \sec^2 x} \Rightarrow \frac{-1}{2}$

$\Rightarrow y = e^{-1/2} \Rightarrow y = 1/\sqrt{e}$

Q2) $\lim_{x\to 0} x^x$

Sol^n let $y = \lim_{x\to 0} x^x$

applying log both side

$\Rightarrow \log y = \lim_{x\to 0} \log x^x \Rightarrow \log y = \lim_{x\to 0} x \log x$

$\Rightarrow \log y = \lim_{x\to 0} \frac{\log x}{1/x}$ {Now this is ∞/∞ form}

$\Rightarrow \log y = \lim_{x\to 0} \frac{\frac{1}{x}}{-1/x^2}$ (applying LH rule)

$\Rightarrow \log y = \lim_{x\to 0} -x \quad \Rightarrow \log y = 0$

$\Rightarrow y = e^0 \quad \Rightarrow y = 1$ ans//

Q3) evaluate $\lim_{x\to\pi/2} (\sin x)^{\tan x}$

Soln this is 1^∞ form

Soln $\log y = \lim_{x\to\pi/2} \log(\sin x)^{\tan x}$

$\Rightarrow \log y = \lim_{x\to\pi/2} \tan x \log \sin x$

$\Rightarrow \log y = \lim_{x\to\pi/2} \frac{\log \sin x}{\cot x}$ this is 0/0 form

applying LH rule

$\Rightarrow \log y = \lim_{x\to\pi/2} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$

$\Rightarrow \log y = \lim_{x\to\pi/2} \frac{\cos x/\sin x}{-1/\sin^2 x}$

$\Rightarrow \log y = \lim_{x\to\pi/2} -\cos x \times \sin x$

$\Rightarrow \log y = 0 \quad \Rightarrow y = e^0 \Rightarrow y = 1$ ans

Q4) evaluate $\lim_{x\to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$

Soln this is 1^∞ form

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \log\left(\frac{\sinh x}{x}\right)$

$\left\{ \because \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} \right\}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \log \left[\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots}{x}\right]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \log \frac{x\left[1 + x^2/6 + x^4/5!\right]}{x}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \log[1+z]$

where $z = x^2/6 + x^4/5! + \dots$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} [z - z^2/2 + \dots]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \left[\frac{x^2}{6} + \frac{x^4}{120} + \dots\right] - \frac{1}{2}\left[\frac{x^2}{6} + \frac{x^4}{120} + \dots\right]^2$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \left[\frac{x^2}{6} + \text{other terms}\right]$

$\Rightarrow \log y = 1/6 \quad \Rightarrow y = e^{1/6}$

Q5) evaluate $\lim_{x\to\infty} \left[\frac{\log x}{x}\right]^{1/x}$

Soln so this is ∞^0 form

$\log y = \lim_{x\to\infty} \frac{1}{x} \log\left[\frac{\log x}{x}\right]$

$\log y = \lim_{x\to\infty} \frac{1}{x} [\log(\log x) - \log x]$

$\log y = \lim_{x\to\infty} \frac{\log(\log x)}{x} - \lim_{x\to\infty} \frac{\log x}{x}$

applying LH rule

$\Rightarrow \log y = \lim_{x\to\infty} \frac{1}{\log x} \cdot \frac{1}{x} - \lim_{x\to\infty} \frac{1}{x}$

$\Rightarrow \log y = 0 - 0$

$\Rightarrow y = e^0 \Rightarrow y = 1$

Q6) evaluate $\lim_{x\to 0} \left[\frac{\tan x}{x}\right]^{1/x}$

Soln this is 1^∞ form

$\log y = \lim_{x\to 0} \frac{1}{x} \log\left[\frac{\tan x}{x}\right]$

$\left\{ \because \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \right\}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x} \log \frac{[x + x^3/3 + 2x^5/15 + \cdots]}{x}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x} \log \frac{\cancel{x}[1 + x^2/3 + 2x^4/15 + \cdots]}{\cancel{x}}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x} \log(1+z)$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x} [z - z^2/2 + \cdots]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x} \left[(x^2/3 + 2x^4/15) - \frac{1}{2}\left[\frac{x^2}{3} + \frac{2x^4}{15}\right]^2\right]$

$\Rightarrow \log y = \lim_{x\to 0} \left[\frac{x}{3} + \frac{2x^3}{15} \cdots\right]$

$\Rightarrow \log y = 0 \Rightarrow y = e^0$

$\Rightarrow y = 1$

Q7) evaluate $\lim_{x\to 0} (\cos x)^{1/x^2}$

Soln this is 1^∞ form

$\log y = \lim_{x\to 0} \frac{1}{x^2} \log(\cos x)$ (0/0 form)

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{2x} \frac{-\sin x}{\cos x}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{-\sin x}{2x \cos x} \Rightarrow \log y = \lim_{x\to 0} \frac{-\tan x}{2x}$

again by LH rule

$\Rightarrow \log y = \lim_{x\to 0} -\frac{\sec^2 x}{2} \Rightarrow \log y = \frac{-1}{2}$

$\Rightarrow y = e^{-1/2} \Rightarrow y = 1/\sqrt{e}$

Q8) $\lim_{x\to 1} (1-x^2)^{1/\log(1-x)}$

Soln this is 0^0 form

$\log y = \lim_{x\to 1} \frac{1}{\log(1-x)} \log(1-x^2)$

Now it is ∞/∞ form

applying LH rule

$\Rightarrow \log y = \lim_{x\to 1} \frac{\frac{-2x}{(1-x^2)}}{\frac{-1}{(1-x)}}$ $\Rightarrow$ ~~$\log y = \lim_{x\to 1} \frac{2x}{(1-x^2)(1-x)}$~~

$\Rightarrow \log y = \lim_{x\to 1} \frac{2x}{(1+x)\cancel{(1-x)}} \times \frac{\cancel{(1-x)}}{1}$

$\Rightarrow \log y = \lim_{x\to 1} = \frac{2x}{(1+x)} \Rightarrow \log y = \frac{2 \cdot 1}{x}$

$\Rightarrow y = e^1 \Rightarrow y = e$ ans

Q9) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

Sol^n $\log y = \lim_{x\to 0} \frac{1}{x^2} \log\left[\frac{\tan x}{x}\right]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \log\left[\frac{x + x^3/3 + 2x^5/15 + \dots}{x}\right]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \log \frac{\cancel{x}\,[1 + x^2/3 + 2x^4/15 + \dots]}{\cancel{x}}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \log[1+z]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} [z - z^2/2]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x^2} \left[\left(\frac{x^2}{3} + \frac{2x^4}{15} + \dots\right) - \left(\frac{x^2}{3} + \frac{2x^4}{15} + \dots\right)^2\right]$

$\Rightarrow \log y = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}} \left[\frac{1}{3} + \frac{2x^2}{15} + \dots\right]$

$\Rightarrow \log y = \frac{1}{3} \Rightarrow y = e^{1/3}$ ans

Q10) $\lim_{x\to 0} \sin x \cdot \log x$ [this is $0 \times \infty$ form]

Sol^n $\lim_{x\to 0} \frac{\log x}{1/\sin x}$ now this is ∞/∞ form

$\Rightarrow \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x}$

applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{\frac{1}{x}}{-\cot x \operatorname{cosec} x}$

$\lim_{x\to 0} \frac{\sin x}{1/\log x}$ now this is 0/0 form

applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{\cancel{x}\cos x}{1/1/x} \Rightarrow \lim_{x\to 0} \frac{\cos x}{x}$

again applying LH rule

$\Rightarrow \lim_{x\to 0} \frac{-\sin x}{1} \Rightarrow 0$ // ans

Q10) find the values of a, b and c so that $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$

Sol^n $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

$\Rightarrow \lim_{x\to 0} a\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right] - b\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right] +$

$$\frac{c\left[1 - x + \frac{x^2}{2!}\dots\right]}{x\left[x - \frac{x^3}{3!} + \dots\right]}$$

$$\Rightarrow \lim_{x\to 0} \frac{\left[(a-b+c) + (a-c)x + \frac{(a+b+c)x^2}{2} + \dots\right]}{x\left[x - \frac{x^3}{3!} + \dots\right]}$$

$\therefore$ (1) $a-b+c=0$, $a-c=0$, $\frac{a+b+c}{2} = 2$ -

$\Rightarrow a=c$ - (2)

$\Rightarrow a+b+c=4$ - (3)

$\Rightarrow 2a+b=4$ - (4) (Putting $a=c$ in eq^n 3)

Putting $a=c$ in eq^n (1)

$\Rightarrow 2a-b=0$ - (5)

Adding (4) and (5)

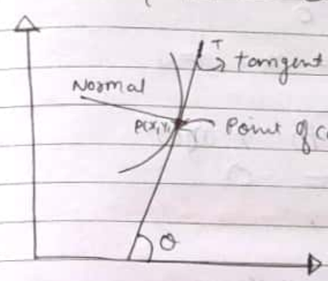
$$\begin{aligned} 2a+b &= 4 \\ 2a-b &= 0 \\ \hline 4a &= 4 \Rightarrow a = 1 = c \end{aligned}$$

$\Rightarrow 1-b+1=0 \Rightarrow b=2$

so $a=1$, $b=2$ & $c=1$

Unit 2

TANGENT & NORMAL



$\Rightarrow$ Slope of tangent $\Rightarrow m = \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$
at Point $P(x_1,y_1)$

$\Rightarrow$ Slope of Normal $\Rightarrow -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}$
at Point $P(x_1,y_1)$

$\Rightarrow$ eq^n of tangent at Point $P(x_1,y_1)$:-

$$(y-y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x-x_1)$$

$\Rightarrow$ eq^n of normal at Point $P(x_1,y_1)$:-

$$(y-y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x-x_1)$$

$\Rightarrow$ Some Imp points

- If tangent is ∥ to x-axis, then $\theta=0$
  slope of tangent $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$
  eq^n of tangent :- $y-y_1=0$

- If tangent is $\perp$ to x-axis then $\theta = 90°$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{1}{0} = \infty$$

eq^n of tangent $= x - x_1 = 0$

- If Normal is $\parallel$ to x axis

slope of Normal $= -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} = 0$

eq^n of normal :- $y - y_1 = 0$

- If normal is $\perp$ to x axis

slope of normal $= -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} = \frac{1}{0} = \infty$

eq^n of normal :- $x - x_1 = 0$

Q1) In the given curve $3b^2y = x^3 - 3ax^2$ find the points at which the tangent is $\parallel$ to x axis

Sol^n eq^n of curve :- $3b^2y = x^3 - 3ax^2$ —①

diff. w.r.to x :-

$$3b^2\frac{dy}{dx} = 3x^2 - 6ax$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 6ax}{3b^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 2ax}{b^2} \text{ —② } \left[\text{slope of tangent at } P(x,y)\right]$$

If tangent is $\parallel$ to x axis then $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{x^2 - 2ax}{b^2} = 0 \Rightarrow x^2 - 2ax = 0$$

$$\Rightarrow x(x - 2a) = 0 \Rightarrow x = 0 \ \& \ x = 2a$$

- If $x = 0$ (Putting this in eq^n (1))

$$y = \frac{0 - 0}{3b^2} \Rightarrow y = 0$$

$(0,0)$

- If $x = 2a$, $y = \frac{8a^3 - 3a \times 4a^2}{3b^2}$

$$\Rightarrow y = \frac{8a^3 - 12a^3}{3b^2} = \frac{-4a^3}{3b^2}$$

$(2a, -4a^3/3b^2)$

$\Rightarrow$ The required points are $(0,0)$ & $\left(2a, -\frac{4a^3}{3b^2}\right)$

Q2) At what point on curve $y = \sin x$ is the tangent $\parallel$ to the x axis

eq^n of curve :- $y = \sin x$ —①

diff. w.r.to x

$$\frac{dy}{dx} = \cos x \quad \left[\text{slope of tangent at } P(x,y)\right]$$

If the tangent is $\parallel$ to x then $dy/dx = 0$

$\Rightarrow \cos x = 0$ $\left[\because \cos\theta = 0 \text{ then } \theta = (2n+1)\pi/2, n \in Z\right]$

$\Rightarrow x = (2n+1)\pi/2$

Put this value in eq^n ①

$$\Rightarrow y = \sin\left[(2n+1)\frac{\pi}{2}\right] \Rightarrow y = \sin\left[n\pi + \frac{\pi}{2}\right]$$

$\{\because \sin(n\pi+\theta) = (-1)^n \sin\theta\}$

$\Rightarrow y = (-1)^n \sin \pi/2$

$\Rightarrow y = (-1)^n$

Required point :- $\left[(2n+1)\frac{\pi}{2}, (-1)^n\right]$

Q3) Find the eqn of tangent at the point $(2,2)$ on the curve. $x^2 - 2xy + y^2 + 2x + y - 6 = 0$

Soln eqn of curve $= x^2 - 2xy + y^2 + 2x + y - 6 = 0$

diff. w r to x, we get

$\Rightarrow 2x - 2\left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} + 2 + \frac{dy}{dx} - 0 = 0$

$\Rightarrow 2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} + 2 + \frac{dy}{dx} = 0$

$\Rightarrow (2x - 2y - 1)\frac{dy}{dx} = 2x - 2y + 2$

$\Rightarrow \frac{dy}{dx} = \frac{2x - 2y + 2}{2x - 2y - 1}$

$\Rightarrow \left[\frac{dy}{dx}\right]_{(2,2)} = \frac{2.2 - 2.2 + 2}{2.2 - 2.2 - 1} = \frac{4 - 4 + 2}{4 - 4 - 1} = -2$

Hence the required eqn of tangent at the point $(2,2)$

$\Rightarrow (y - y_1) = m(x - x_1)$

$\Rightarrow (y-2) = -2(x-2)$

$\Rightarrow y - 2 = -2x + 4 \Rightarrow 2x + y - 6 = 0$ ans

Q4) Find the equation of tangent at the point $(a\cos\theta, b\sin\theta)$ on the curve $(x^2/a^2) + (y^2/b^2) = 1$

Soln eqn of curve $\Rightarrow (x^2/a^2) + (y^2/b^2) = 1$

diff. w. r. to x

$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y} \Rightarrow \frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$

at the point $(a\cos\theta, b\sin\theta)$

$\Rightarrow \frac{dy}{dx} = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta} \Rightarrow \frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$

Hence the required eqn of tangent at the point $(a\cos\theta, b\sin\theta)$ is

$\Rightarrow (y - b\sin\theta) = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$\Rightarrow ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

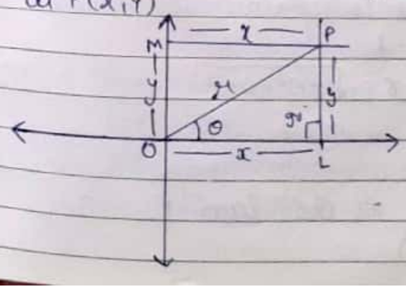
$\Rightarrow ay\sin\theta + bx\cos\theta = ab\cos^2\theta + ab\sin^2\theta$

$\Rightarrow ay\sin\theta + bx\cos\theta = ab(\cos^2\theta + \sin^2\theta)$

$\Rightarrow ay\sin\theta + bx\cos\theta = ab$ ans

Polar Co-ordinate System

at $P(x,y)$



$\Rightarrow \cos\theta = \frac{x}{r}$

$\Rightarrow \boxed{r\cos\theta = x}$ —(1)

$\Rightarrow \sin\theta = \frac{y}{r}$

$\Rightarrow \boxed{r\sin\theta = y}$ —(2)

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$\Rightarrow P(x,y) = P(r\cos\theta, r\sin\theta)$
$= f(r,\theta)$ or
$= P(r,\theta)$ where r,θ are the polar coordinate of P.

from eq^n ① & ②
Squaring both the eq^n & adding them

$\Rightarrow x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$
$\Rightarrow x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$\Rightarrow x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$
$\Rightarrow x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

Now, eq^n ② $\div$ eq^n ①

$\Rightarrow \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \Rightarrow \tan\theta = \frac{y}{x}$

$\Rightarrow \boxed{\theta = \tan^{-1}(y/x)}$

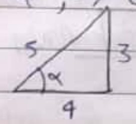
$\because OP = r$ = radius vector (Thus can't be -ve)
O = Pole
$\angle POX = \theta$ = Vectorial angle
OX = initial line

Ques1) Convert $(-1,-1)$ into polar coordinate
Sol: $\because x = -1, y = -1$
$r = \sqrt{x^2+y^2} \Rightarrow r = \sqrt{(-1)^2+(-1)^2}$
$\Rightarrow \boxed{r = \sqrt{2}}$
$\{\because \theta = \tan^{-1}(y/x)\}$
$\Rightarrow \theta = \tan^{-1}(-1/-1) \Rightarrow \theta = \tan^{-1} 1$
$\Rightarrow \theta = \tan^{-1}(\tan \pi/4)$

$\Rightarrow \boxed{\theta = \pi/4}$
Since $(-1,-1)$ lies in IIIrd Quadrant
So $\theta = \pi + \frac{\pi}{4}$ i.e $\frac{5\pi}{4}$

$\boxed{\text{Correct ans} = (\sqrt{2}, 5\pi/4)}$

Ques2) Convert $(5, \tan^{-1}(3/4) - \pi)$ into Cartesian co-ordinate
Sol: $r = 5$, $\theta = \tan^{-1}(3/4) - \pi$
let $\tan^{-1}(3/4) = \alpha$, $\Rightarrow \theta = \alpha - \pi$
$\Rightarrow \tan\alpha = 3/4$
$\Rightarrow \sin\alpha = 3/5$
$\Rightarrow \cos\alpha = 4/5$



$\because x = r\cos\theta \Rightarrow x = 5\cos(\alpha - \pi)$
$\Rightarrow x = 5\cos(\pi - \alpha) \Rightarrow x = -5\cos\alpha$ $\{\because \cos(\pi-\theta) = -\cos\theta\}$
$\Rightarrow x = -\cancel{5} \times \frac{4}{\cancel{5}} \Rightarrow \boxed{x = -4}$

$\because y = r\sin\theta \Rightarrow y = 5\sin(\alpha - \pi)$
$\Rightarrow y = 5\sin(\pi - \alpha) \Rightarrow y = -5\sin\alpha$
$\Rightarrow y = -\cancel{5} \times \frac{3}{\cancel{5}} \Rightarrow \boxed{y = -3}$

So Cartesian coordinat will be $(-4,-3)$

Ques3) Transform the eq^n $r^2\cos 2\theta = a^2$ to Cartesian form
Sol: $r^2\cos 2\theta = a^2$
we know that $r^2 = x^2+y^2$ & $\tan\theta = \frac{y}{x}$

$\Rightarrow \left\{ \because \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} \right\}$

$\Rightarrow r^2\left[\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right] = a^2$

$\Rightarrow (x^2+y^2)\left[\dfrac{1-\frac{y^2}{x^2}}{1+\frac{y^2}{x^2}}\right] = a^2$

$\Rightarrow (x^2+y^2)\left[\dfrac{x^2-y^2}{x^2+y^2}\right] = a^2$

$\Rightarrow \boxed{x^2-y^2=a^2}$ ans//

Ques 1) Transform the eqn $y = x\tan\alpha$ to polar form

Sol^n $y = x\tan\alpha$

We know that $x = r\cos\theta$ & $y = r\sin\theta$

$\Rightarrow r\sin\theta = r\cos\theta \tan\alpha$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} = \tan\alpha \Rightarrow \tan\theta = \tan\alpha$

$\Rightarrow \boxed{\theta = \alpha}$ this is the polar form

Ques) Convert $2x - 5x^3 = 11xy$ into polar coordinate

Sol^n $2x - 5x^3 = 11xy$

$\{\because x = r\cos\theta,\ y = r\sin\theta\}$

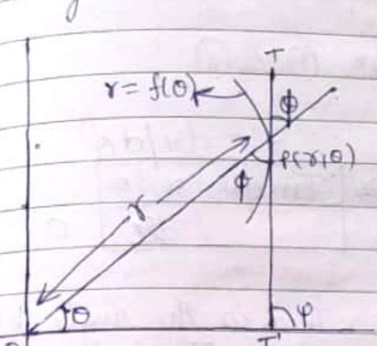
$\Rightarrow 2r\cos\theta - 5r^3\cos^3\theta = 11\, r\cos\theta \cdot r\sin\theta$

$\Rightarrow 2r\cos\theta - 5r^3\cos^3\theta = 11\, r^2\cos\theta\sin\theta$

$\Rightarrow 2\cos\theta - 5r^2\cos^3\theta = 11 r\cos\theta\sin\theta$

$\Rightarrow 5r^2\cos^3\theta + 11\, r\cos\theta\sin\theta - 2\cos\theta = 0$

Angle between radius vector and tangent



Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$

Let $T'PT$ be the tangent to the curve, $\angle OPT' = \phi$

from fig. $\psi = \theta + \phi$

$\Rightarrow \tan\psi = \tan[\theta + \phi]$

$\Rightarrow \tan\psi = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi} = \dfrac{dy}{dx}$ — (1)

$\because (x, y)$ is a cartesian coordinate of Point P

$x = r\cos\theta$, $y = r\sin\theta$

Since r is a function of θ

$\Rightarrow \dfrac{dx}{d\theta} = -r\sin\theta + r'\cos\theta$

$\Rightarrow \dfrac{dy}{d\theta} = r\cos\theta + r'\sin\theta$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$

$\Rightarrow \tan\psi = \dfrac{\cancel{r'\cos\theta}\left[\frac{r}{r'} + \frac{\sin\theta}{\cos\theta}\right]}{\cancel{r'\cos\theta}\left[1 - \frac{r}{r'}\frac{\sin\theta}{\cos\theta}\right]}$

$$\Rightarrow \tan\psi = \frac{\tan\theta + \frac{r}{r'}}{1 - \frac{r}{r'}\tan\theta} \quad \text{--- (2)}$$

On comparing eqⁿ (1) & (2)

$\Rightarrow \tan\phi = r/r' \quad \because r' = dr/d\theta$

$\Rightarrow \tan\phi = \frac{r}{\frac{dr}{d\theta}} \Rightarrow \boxed{\tan\phi = \frac{r\,d\theta}{dr}}$ or

$\Rightarrow \phi = \tan^{-1}\left[\frac{r\,d\theta}{dr}\right]$ This is the angle b/w radius vector & tangent

Ques) Find the angle b/w the radius vector & the tangent at any point on the cardioid, $r = a(1-\cos\theta)$

Sol:ⁿ eqⁿ of curve $= r = a(1-\cos\theta)$

$\Rightarrow \frac{dr}{d\theta} = a\sin\theta \Rightarrow \frac{d\theta}{dr} = \frac{1}{a\sin\theta}$

$\Rightarrow \tan\phi = r\cdot\frac{d\theta}{dr} \Rightarrow \tan\phi = \cancel{a}(1-\cos\theta)\frac{1}{\cancel{a}\sin\theta}$

$\Rightarrow \left\{\because \cos\theta = 1 - 2\sin^2\theta/2 \atop \sin\theta = 2\sin\theta/2\cdot\cos\theta/2\right\}$

$\Rightarrow \tan\psi = \frac{1-(1-2\sin^2\theta/2)}{2\sin\theta/2\cos\theta/2} \Rightarrow \tan\psi = \frac{\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2}\,\cancel{\sin\theta/2}\cos\theta/2}$

$\Rightarrow \tan\psi = \frac{\sin\theta/2}{\cos\theta/2} \Rightarrow \tan\psi = \tan\theta/2$

$\Rightarrow \boxed{\psi = \theta/2}$ ans//

Ques) Find the eqⁿ of the tangent at the point (x, y) on each of the curve $(x^2+y^2)^2 = a^2(x^2-y^2)$

Solⁿ eqⁿ of curve :- $(x^2+y^2)^2 = a^2(x^2-y^2)$

diff. w. r. to x

$$\Rightarrow 2(x^2+y^2)\left[2x + 2y\frac{dy}{dx}\right] = a^2\left[2x - 2y\frac{dy}{dx}\right]$$

$$\Rightarrow 4x(x^2+y^2) + 4y\frac{dy}{dx}(x^2+y^2) = a^2 2x - 2a^2y\frac{dy}{dx}$$

$$\Rightarrow [4y(x^2+a^2) + 2a^2y]\frac{dy}{dx} = 2xa^2 - 4x(x^2+y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{xa^2 - 2x(x^2+y^2)}{2y(x^2+a^2)+a^2y}$$

$$\Rightarrow \boxed{\left[\frac{dy}{dx}\right]_{(x,y)} = \frac{xa^2 - 2x(x^2+y^2)}{2y(x^2+a^2)+a^2y}} \text{ slope of tangent (m)}$$

$\{\because (Y - Y_1) = m(X - X_1)\}$

$$\Rightarrow Y - y = \frac{xa^2 - 2x(x^2+y^2)}{2y(x^2+y^2)+a^2y}(X - x)$$

$$\Rightarrow Y[2y(x^2+y^2)+a^2y] - y[2y(x^2+y^2)+a^2y] = X[xa^2 - 2x(x^2+y^2)] - x[xa^2 - 2x(x^2+y^2)]$$

$$\Rightarrow Y[2y(x^2+y^2)+a^2y] - 2y^2(x^2+y^2) - a^2y^2 = X[xa^2 - 2x(x^2+y^2)] - x^2a^2 + 2x^2(x^2+y^2)$$

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$\Rightarrow Y[2y(x^2+y^2)+a^2y] - X[xa^2 - 2x(x^2+y^2)] = 2y^2(x^2+y^2)+a^2y^2 - x^2a^2 + 2x^2(x^2+y^2)$

$\Rightarrow Y[2y(x^2+y^2)+a^2y] - X[xa^2-2x(x^2+y^2)] = 2(x^2+y^2)[y^2+x^2] - a^2[x^2-y^2]$

$\{\because (x^2+y^2)^2 = a^2(x^2-y^2)\}$

$\Rightarrow Y[2y(x^2+y^2)+a^2y] - X[xa^2 - 2x(x^2+y^2)] = 2(x^2+y^2)^2 - a^2(x^2-y^2)$

$\Rightarrow Y[2y(x^2+y^2)+a^2y] - X[xa^2 - 2x(x^2+y^2)] = 2a^2(x^2-y^2) - a^2(x^2-y^2)$

$\Rightarrow Y[2y(x^2+y^2)+a^2y] - X[xa^2 - 2x(x^2+y^2)] = a^2(x^2-y^2)$ ans//

Ques2) find the eq^n of tangent at the point t on each of the following curve :-

$x = a(t+\sin t)$, $y = a(1-\cos t)$

Sol^n $\frac{dx}{dt} = a(1+\cos t)$ $\quad \frac{dy}{dt} = a\sin t$

$\Rightarrow \frac{dy}{dx} = \frac{\cancel{a}\sin t}{\cancel{a}(1+\cos t)}$ $\left\{ \begin{array}{l} \because \sin t = 2\sin\theta/2 \cdot \cos\theta/2 \\ 1+\cos t = 2\cos^2 t/2 \end{array} \right.$

$\Rightarrow \frac{dy}{dx} = \frac{\cancel{2}\sin t/2 \cdot \cancel{\cos t/2}}{\cancel{2}\cos^{\cancel{2}} t/2}$

$\Rightarrow \boxed{\left.\frac{dy}{dx}\right|_{x=t}} = \tan t/2$

$\{\because Y - Y_1 = m(x-x_1)\}$

$\Rightarrow Y - [a(1-\cos t)] = \tan t/2\,(X - (a(t+\sin t)))$ — ans//

$\Rightarrow Y - a \times 2\sin^2 t/2 = \cancel{X}\tan t/2 - a\tan t/2\,(t+\sin t)$

$\Rightarrow Y - 2a\sin^2 t/2 = X\tan t/2 - at\tan t/2 - a\tan t/2 \sin t/2$

$\Rightarrow Y - 2a\sin^2 t/2 = \tan t/2\,[X - at] - a\frac{\sin t/2}{\cos t/2} \times 2\sin t/2\cos t/2$

$\Rightarrow Y - \cancel{2a\sin^2 t/2} = \tan t/2\,[X - at] - \cancel{2a\sin^2 t/2}$

$\Rightarrow Y = [X - at]\tan t/2$ ans//

Ques3) find the points at which the tangent is (a) $\parallel$ to (b) $\perp$ to the X axis of the curve $ax^2+2hxy+by^2=1$

Sol^n eq^n of curve $= ax^2+2hxy+by^2 = 1$ — (1)

diff. w. r. to x

$\Rightarrow 2ax + 2hx\cdot\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$

$\Rightarrow (2ax+2hy) + [2hx+2by]\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-(2ax+2hy)}{2hx+2by}$ $\Rightarrow \frac{dy}{dx} = -\frac{(ax+hy)}{hx+by} = m$

when tangent is $\parallel$ to x axis then $\frac{dy}{dx} = 0$

$\Rightarrow \frac{-(ax+hy)}{hx+by} = 0$ $\Rightarrow \boxed{ax+hy=0}$ — (2)

when tangent is $\perp$ to x axis then $\frac{dy}{dx} = \frac{1}{0}$

or $\frac{dx}{dy} = 0$

$\Rightarrow -\frac{(ax+hy)}{(hx+by)} = \frac{1}{0} \Rightarrow \boxed{hx+by=0}$ — ③

from eq^n ② :- i.e $ax+hy=0$

$\Rightarrow x = \frac{-hy}{a}$ put this in eq^n ①

$\Rightarrow a \times \left[\frac{-hy}{a}\right]^2 + 2h\left[\frac{-hy}{a}\right]y + by^2 = 0$

$\Rightarrow a \times \frac{h^2y^2}{a^2} - \frac{2h^2y^2}{a} + by^2 = 0$

$\Rightarrow \frac{h^2y^2 - 2h^2y^2}{a} + by^2 = 0$

$\Rightarrow by^2 - \frac{h^2y^2}{a} = 0 \Rightarrow aby^2 - h^2y^2 = 0$

$\Rightarrow y^2[ab - h^2] = 0 \Rightarrow y = 0$

Point at which the tangent is ∥ is:- $[-hy/a, 0]$

from eq^n ③ i.e $hx+by=0$

$\Rightarrow x = \frac{-by}{h}$ put this in eq^n ①

$\Rightarrow a \times \left[\frac{-by}{h}\right]^2 + 2h\left[\frac{-by}{h}\right]y + by^2 = 0$

$\Rightarrow \frac{ab^2y^2}{h^2} - \frac{2\cancel{h}by^2}{\cancel{h}} + by^2 = 0$

$\Rightarrow \frac{ab^2y^2}{h^2} - by^2 = 0 \Rightarrow ab^2y^2 - hby^2 = 0$

$\Rightarrow by^2[ab - h] = 0 \Rightarrow y = 0$

Point at which the tangent is ⊥ is:- $[-by/h, 0]$

Req. points are:- $\left[\frac{-hy}{a}, 0\right]$ & $\left[\frac{-by}{h}, 0\right]$

Ques 4) find the angle of intersection of curves $xy = a^2$ and $x^2+y^2 = 2a^2$

Sol^n given curves $\Rightarrow xy = a^2$ — ①

$x^2+y^2 = 2a^2$ — ②

$\Rightarrow$ from ① :- $y = a^2/x$

putting the value of $y = a^2/x$ in eq^n ②

$\Rightarrow x^2 + \frac{a^4}{x^2} = 2a^2 \Rightarrow x^4 + a^4 = 2a^2x^2$

$\Rightarrow (x^2)^2 + (a^2)^2 - 2a^2x^2 = 0$

$\Rightarrow (x^2 - a^2)^2 = 0 \Rightarrow \boxed{x = \pm a}$

$\Rightarrow$ when $x = a$ (putting in eq^n ①)

then $y = a$

Point P(a, a)

$\Rightarrow$ when $x = -a$, then $y = -a$

point Q$(-a, -a)$

Since both curves are intersecting at point P(a,a) & Q$(-a,-a)$

$\Rightarrow$ from eq^n ① :- $x\frac{dy}{dx} + y = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(a,a)} = -\frac{\cancel{a}}{\cancel{a}} = -1 = m_1$$

$\Rightarrow$ from eq^n ②:- $2x + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = \frac{-x}{y}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(-a,-a)} = \frac{-(-a)}{-a} = -1 = m_2$$

When $m_1 = m_2$ then $\theta = 0^\circ$, curves touches at point p.

Ques) Show that the condition that the curve $ax^2 + by^2 = 1$ & $a'x^2 + b'y^2 = 1$ do intercept orthogonally is that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

Sol^n eq^n of curves:- $ax^2 + by^2 = 1$ —① & $a'x^2 + b'y^2 = 1$ —②

diff. w. r. to x eq^n ①

$$\Rightarrow 2ax + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\cancel{2}ax}{\cancel{2}by} = \frac{-ax}{by}$$

let both the curve intersect at point $P(x_1, y_1)$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{ax_1}{by_1} = m_1$$

from eq^{n}② diff. w. r. to x

$$\Rightarrow 2a'x + 2b'y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-a'x}{b'y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-a'x_1}{b'y_1}$$

{ for orthogonally intersection:- $m_1 m_2 = -1$ }

$$\Rightarrow \frac{-ax_1}{by_1} \times \left(\frac{-a'x_1}{b'y_1}\right) = -1$$

$$\Rightarrow \frac{aa'x_1^2}{bb'y_1^2} = -1 \quad \Rightarrow \frac{x_1^2}{y_1^2} = \frac{-bb'}{aa'}$$

both eq^n of curve can be written as:-

$ax_1^2 + by_1^2 = 1$ & $a'x_1^2 + b'y_1^2 = 1$

$$\Rightarrow ax_1^2 + by_1^2 = a'x_1^2 + b'y_1^2$$

$$\Rightarrow ax_1^2 - a'x_1^2 = b'y_1^2 - by_1^2$$

$$\Rightarrow (a - a')x_1^2 = y_1^2(b' - b)$$

$$\Rightarrow \boxed{\frac{x_1^2}{y_1^2} = \frac{b' - b}{a - a'}} \quad — ③$$

from eq^n ③

$$\Rightarrow \frac{b' - b}{a - a'} = \frac{-bb'}{aa'} \Rightarrow \frac{b' - b}{bb'} = \frac{-(a - a')}{aa'}$$

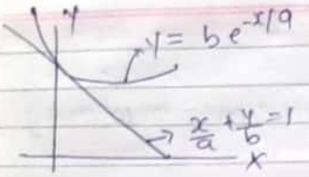
$$\Rightarrow \frac{\cancel{b'}}{b\cancel{b'}} - \frac{\cancel{b}}{\cancel{b}b'} = \frac{-\cancel{a}}{\cancel{a}a'} + \frac{\cancel{a'}}{a\cancel{a'}}$$

$$\Rightarrow \frac{1}{b} - \frac{1}{b'} = -\frac{1}{a'} + \frac{1}{a}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$$

Hence Proved

Ques6) Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of Y



1) eqⁿ of curve :- $y = be^{-x/a}$
Since the curve cross the y axis then $x=0$
so $y = be^{-0/a} \Rightarrow y = b$
Point $P(0, b)$

diff. eqⁿ (1) w. r. to x

$\Rightarrow \frac{dy}{dx} = -\frac{b}{a} e^{-x/a}$

$\Rightarrow \left[\frac{dy}{dx}\right]_{(0,b)} = \frac{-b}{a}$

$\{\because (y-y_1) = m(x-x_1)\}$

$\Rightarrow y-b = \frac{-b}{a}(x-0) \Rightarrow \frac{y-b}{b} = \frac{-1}{a}x$

$\Rightarrow \frac{y}{b} - 1 = \frac{-x}{a} \Rightarrow \frac{x}{a}+\frac{y}{b} = 1$ Hence Proved

Ques8) Prove that the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a}+\frac{y}{b}=2$ at the point (a,b) whatever be the value of 'n'

Soln eqⁿ of curve :- $\left[\frac{x}{a}\right]^n + \left[\frac{y}{b}\right]^n = 2$

diff. w. r. to x :-

$\Rightarrow n\left[\frac{x}{a}\right]^{n-1}\frac{1}{a} + n\left[\frac{y}{b}\right]^{n-1}\frac{1}{b}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}\frac{1}{b}\left[\frac{y}{b}\right]^{n-1} = -\frac{1}{a}\left[\frac{x}{a}\right]^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{-\frac{1}{a}\left(\frac{x}{a}\right)^{n-1}}{\frac{1}{b}\left[\frac{y}{b}\right]^{n-1}}$ $\Rightarrow \left(\frac{dy}{dx}\right)_{(a,b)} = -\frac{b}{a}$

$\Rightarrow y-y_1 = m(x-x_1)$

$\Rightarrow y-b = \frac{-b}{a}(x-a) \Rightarrow \frac{y}{b} - \cancel{\frac{b}{b}} = \frac{-x}{a} + \cancel{\frac{a}{a}}$

$\Rightarrow \frac{x}{a}+\frac{y}{b} = 2$ Hence Proved

Ques9) Prove that the sum of the intercepts on the co-ordinate axes of any tangent to $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is constant

Soln eqⁿ of curve = $\sqrt{x}+\sqrt{y}=\sqrt{a}$ — (1)

diff. w. r. to x :-

$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}} \Rightarrow \left(\frac{dy}{dx}\right)_{(x,y)} = -\frac{\sqrt{y}}{\sqrt{x}}$

$\{Y - y_1 = m(X-x_1)\}$

$\Rightarrow Y - y = \frac{-\sqrt{y}}{\sqrt{x}}(X-x)$

$\Rightarrow \frac{Y}{\sqrt{y}} - \frac{y}{\sqrt{y}} = -\frac{X}{\sqrt{x}} + \frac{x}{\sqrt{x}}$

$\Rightarrow \frac{Y}{\sqrt{y}} - \sqrt{y} = -\frac{X}{\sqrt{x}} + \sqrt{x}$

$\Rightarrow \frac{X}{\sqrt{x}} + \frac{Y}{\sqrt{y}} = \sqrt{x}+\sqrt{y} \Rightarrow \frac{X}{\sqrt{x}}+\frac{Y}{\sqrt{y}} = \sqrt{a}$

$$\Rightarrow \frac{X}{\sqrt{a}\sqrt{x}} + \frac{Y}{\sqrt{a}\sqrt{y}} = 1$$

Intercept on X axis $= \sqrt{a}\cdot\sqrt{x}$

Intercept on Y axis $= \sqrt{a}\ \sqrt{y}$

$\Rightarrow$ sum of Intercept $= \sqrt{a}\cdot\sqrt{x} + \sqrt{a}\cdot\sqrt{y}$

$= \sqrt{a}\,(\sqrt{x}+\sqrt{y})$

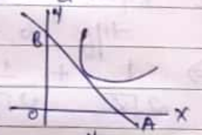
$= \sqrt{a}\cdot\sqrt{a}$

$\Rightarrow$ sum of Intercept $= a$ i.e constant

Ques10) Show that the length of portion of the tangent to the curve $x^{2/3}+y^{2/3} = a^{2/3}$ Intercepted b/w the co-ordinate axes is is constant

Solⁿ eqⁿ of curve $\Rightarrow x^{2/3} + y^{2/3} = a^{2/3}$

diff. w.r. to x :-



$$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = \Rightarrow \left[\frac{dy}{dx}\right]_{(x,y)} = -\frac{y^{1/3}}{x^{1/3}}$$

$\Rightarrow Y - Y_1 = m(X - x_1)$

$$\Rightarrow Y - y = \frac{-y^{1/3}}{x^{1/3}}(X - x)$$

$$\Rightarrow \frac{Y}{y^{1/3}} - \frac{y}{y^{1/3}} = \frac{-X}{x^{1/3}} + \frac{x}{x^{1/3}}$$

$$\Rightarrow \frac{Y}{y^{1/3}} - y^{2/3} = -\frac{X}{x^{1/3}} + x^{2/3}$$

$$\Rightarrow \frac{X}{x^{1/3}} + \frac{Y}{y^{1/3}} = x^{2/3} + y^{2/3}$$

$$\Rightarrow \frac{X}{x^{1/3}} + \frac{Y}{y^{1/3}} = a^{2/3} \Rightarrow \frac{X}{a^{2/3}\cdot x^{1/3}} + \frac{Y}{a^{2/3}\cdot y^{1/3}} = 1$$

Intercept on X-axis $= a^{2/3}\cdot x^{1/3} = OA$

Intercept on Y-axis $= a^{2/3}\cdot y^{1/3} = OB$

$\Rightarrow AB = \sqrt{OA^2 + OB^2}$

$\Rightarrow AB = \sqrt{(a^{2/3}x^{1/3})^2 + (a^{2/3}\cdot y^{1/3})^2}$

$\Rightarrow AB = \sqrt{a^{4/3}x^{2/3} + a^{4/3}\cdot y^{2/3}}$

$\Rightarrow AB = \sqrt{a^{4/3}(x^{2/3}+y^{2/3})}$ $\quad \{\because x^{2/3}+y^{2/3} = a^{2/3}$

$\Rightarrow AB = \sqrt{a^{4/3}\cdot a^{2/3}} = \sqrt{a^{6/3}} = \sqrt{a^2}$

$\Rightarrow AB = a$ = constant ans//

Ques11) If the normal to the curve $x^{2/3}+y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x show that the eqⁿ is $Y\cos\phi - X\sin\phi = a\cos 2\phi$

Solⁿ eqⁿ of curve $= x^{2/3} + y^{2/3} = a^{2/3}$ — (1)

diff. w.r. to x :-

$$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} \Rightarrow \left(\frac{dy}{dx}\right)_{(x,y)} = \frac{-y^{1/3}}{x^{1/3}}$$

$$\Rightarrow \text{slope of normal} = \frac{-1}{\frac{dy}{dx}} = \frac{\mp 1}{\frac{+y^{1/3}}{x^{1/3}}}$$

$$\Rightarrow \text{slope of normal} = \frac{x^{1/3}}{y^{1/3}}$$

$\{\because$ (Given) slope of normal $= \tan\phi$ [because of it is making an angle 'ϕ' $\}$

$$\Rightarrow \tan\phi = \frac{x^{1/3}}{y^{1/3}}$$

$\Rightarrow \dfrac{\sin\phi}{\cos\phi} = \dfrac{x^{1/3}}{y^{1/3}} \Rightarrow \dfrac{y^{1/3}}{\cos\phi} = \dfrac{x^{1/3}}{\sin\phi}$

$\left\{ \because \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}} \right\}$

$\Rightarrow \left\{ \dfrac{\sqrt{y^{2/3}+x^{2/3}}}{\sqrt{\cos^2\phi+\sin^2\phi}} \Rightarrow \sqrt{a^{2/3}} \right\}$

$\Rightarrow \dfrac{y^{1/3}}{\cos\phi} = \dfrac{x^{1/3}}{\sin\phi} = \sqrt{a^{2/3}} \Rightarrow \dfrac{y^{1/3}}{\cos\phi} = \dfrac{x^{1/3}}{\sin\phi} = a^{1/3}$

$\Rightarrow y^{1/3} = a^{1/3}\cos\phi \Rightarrow y = (a^{1/3}\cos\phi)^3$

$\Rightarrow \boxed{y = a\cos^3\phi}$ Similarly $\boxed{x = a\sin^3\phi}$

$\Rightarrow$ eq^n of normal $= (Y - Y_1) = \dfrac{-1}{dY/dx}(X - x_1)$

$\Rightarrow Y - a\cos^3\phi = \dfrac{\sin\phi}{\cos\phi}(X - a\sin^3\phi)$

$\Rightarrow Y\cos\phi - a\cos^4\phi = X\sin\phi - a\sin^4\phi$

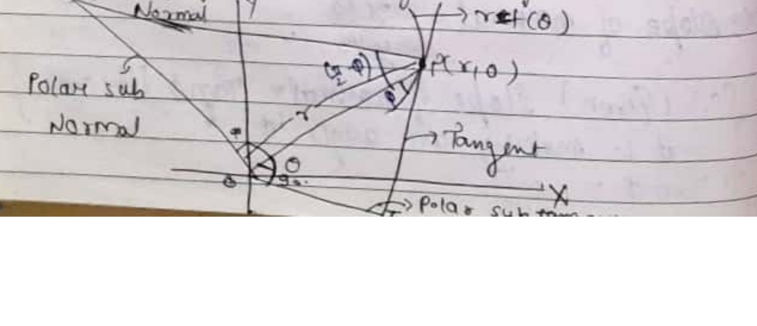
$\Rightarrow Y\cos\phi - X\sin\phi = a(\cos^4\phi - \sin^4\phi)$

$\Rightarrow Y\cos\phi - X\sin\phi = a[(\cos^2\phi - \sin^2\phi)(\cos^2\phi + \sin^2\phi)]$

$\left\{ \begin{array}{l} \because \cos^2\theta - \sin^2\theta = \cos 2\theta \\ \cos^2\theta + \sin^2\theta = 1 \end{array} \right\}$

$\Rightarrow Y\cos\phi - X\sin\phi = a\cos 2\theta$ Hence Proved

Length of polar Subtangent & polar Subnormal



OT = Polar Sub Tangent , ON = Polar sub normal
PT = Polar Tangent , PN = Polar Normal
OP = r

In Δ OPT :- $\tan\phi = \dfrac{OT}{OP} \Rightarrow \tan\phi = \dfrac{OT}{r}$

$\Rightarrow r\tan\phi = OT \Rightarrow r \cdot r\dfrac{d\theta}{dr} = OT$

$\Rightarrow \boxed{\text{Polar Sub Tangent} = r^2\dfrac{d\theta}{dr} = OT}$

In Δ NPO :- $\tan(\pi/2 - \phi) = \dfrac{ON}{OP}$

$\Rightarrow r\cot\phi = ON \Rightarrow \cancel{r} \times \dfrac{1}{\cancel{r}}\dfrac{dr}{d\theta} = ON$

$\Rightarrow \boxed{\text{Polar Sub Normal} = \dfrac{dr}{d\theta}}$

In Δ POT from Pythagoras theoram :-

$\Rightarrow PT = \sqrt{OT^2 + OP^2}$

$\Rightarrow PT \sqrt{\left(r^2\dfrac{d\theta}{dr}\right)^2 + r^2} \Rightarrow \boxed{PT = r\sqrt{r^2\left[\dfrac{d\theta}{dr}\right]^2 + 1}}$

In Δ PON from Pythagoras theoram :-

$\Rightarrow PN = \sqrt{ON^2 + OP^2} \Rightarrow \boxed{PN = \sqrt{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}}$

Ques1) Show that the Curve $r = a\theta$ the polar sub normal is constant and in Curve $r\theta = a$ the polar Subtangent is Constant

eq^n ① :- $r = a\theta \Rightarrow P.S.N = \dfrac{dr}{d\theta} = a$ (constant)

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eqn ② :- $r\theta = a \Rightarrow \theta = a/r$

$\Rightarrow \dfrac{d\theta}{dr} = -\dfrac{a}{r^2} \Rightarrow r^2\dfrac{d\theta}{dr} = -a = PST$

$\Rightarrow PST = r^2\dfrac{d\theta}{dr} = -a$ (constant)

Ques2) find the length of the polar tangent and polar normal of the curve $r = a(1+\cos\theta)$

Sol: eqn of curve $\Rightarrow r = a(1+\cos\theta)$ — ①

diff. w.r.t. to θ :-

$\Rightarrow \dfrac{dr}{d\theta} = -a\sin\theta \Rightarrow \dfrac{d\theta}{dr} = \dfrac{-1}{a\sin\theta}$

$\left\{ \text{So length of Polar tangent :- } r\sqrt{1+\left(r\dfrac{d\theta}{dr}\right)^2} \right\}$

Multiplying by r both side, we get

$\Rightarrow r\dfrac{d\theta}{dr} = \dfrac{-r}{a\sin\theta} \Rightarrow r\dfrac{d\theta}{dr} = -\dfrac{a(1+\cos\theta)}{a\sin\theta}$

$\Rightarrow \left\{ \begin{array}{l} \because 1+\cos\theta = 2\cos^2\theta/2 \\ \sin\theta = 2\sin\theta/2\cos\theta/2 \end{array} \right\}$

$\Rightarrow r\dfrac{d\theta}{dr} = -\dfrac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} \Rightarrow r\dfrac{d\theta}{dr} = -\cot^2\theta/2$

$\Rightarrow \left\{ PT = r\sqrt{1+\left(r\dfrac{d\theta}{dr}\right)^2} \right\} \Rightarrow PT = a(1+\cos\theta)\sqrt{1+\cot^2\theta/2}$

$\Rightarrow PT = a(1+\cos\theta)\times \operatorname{cosec}\theta/2$

$\Rightarrow PT = 2a\cos^2\theta/2 \operatorname{cosec}\theta/2$ ans//

$\left\{ \because PN = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2} \right\}$

$\Rightarrow PN = r\sqrt{1+\left(\dfrac{1}{r^2}\left(\dfrac{dr}{d\theta}\right)^2\right)}$

$\Rightarrow PN = r\sqrt{1+\left[\dfrac{1}{r}\dfrac{dr}{d\theta}\right]^2}$ $\left\{ \because r\dfrac{d\theta}{dr} = \cot^2\theta/2 \Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{1}{\cot^2\theta/2} = \tan^2\theta/2 \right\}$

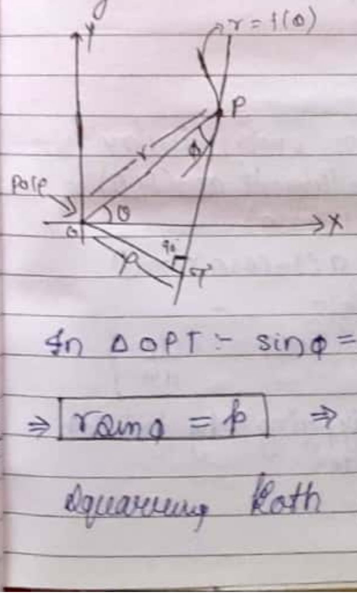
$\Rightarrow PN = \sqrt{1+\tan^2\theta/2}$

$\Rightarrow PN = a(1+\cos\theta)\cdot\sec\theta/2$ $\{\because \sec\theta = 1/\cos\theta\}$

$\Rightarrow PN = 2a\cos^2\theta/2\sec\theta/2$ ans//

$\Rightarrow PN = 2a\cos\theta/2$

Length of perpendicular from pole to the tangent



Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$. Draw OT perpendicular to tangent let $OT = p$

In ΔOPT :- $\sin\phi = \dfrac{OT}{OP} = \dfrac{p}{r}$

$\Rightarrow \boxed{r\sin\phi = p} \Rightarrow \dfrac{1}{p} = \dfrac{1}{r}\operatorname{cosec}\phi$

Squaring both side :-

$\Rightarrow 1/p^2 = 1/r^2 \operatorname{cosec}^2\phi$

$\Rightarrow 1/p^2 = 1/r^2 (1+\cot^2\phi)$ $\quad \left\{ \because \tan\phi = r\dfrac{d\theta}{dr} \Rightarrow \cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta} \right.$

$\Rightarrow \dfrac{1}{p^2} = \dfrac{1}{r^2}\left[1+\left(\dfrac{1}{r}\dfrac{dr}{d\theta}\right)^2\right]$

$\Rightarrow \boxed{\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left[\dfrac{dr}{d\theta}\right]^2}$ — (1)

Length of ⊥ from pole to tangent

Let $u = 1/r$ — (2)

$\Rightarrow$ diff. w. r. to θ :-

$\Rightarrow \dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$ squaring Both sides

$\Rightarrow \left(\dfrac{du}{d\theta}\right)^2 = \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ — (3)

Putting the value of eqⁿ (2) & (3) in eqⁿ (1)

$\Rightarrow \dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$

Ques-1 find the length of perpendicular from the pole on the tangent at (r,θ) of the curve $r = a(1-\cos\theta)$

Solⁿ eqⁿ of curve $= r = a(1-\cos\theta)$ — (1)

diff. w. r. to 'θ' :-

$\Rightarrow \dfrac{dr}{d\theta} = a\sin\theta$ $\quad \left\{\because \tan\phi = r\dfrac{d\theta}{dr}\right\}$

$\Rightarrow \dfrac{d\theta}{dr} = \dfrac{1}{a\sin\theta}$ Multyplying by 'r' Both sides

$\Rightarrow r\dfrac{d\theta}{dr} = \dfrac{r}{a\sin\theta}$

$\Rightarrow \tan\phi = \dfrac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta} = \dfrac{1-[1-2\sin^2\theta/2]}{2\sin\theta/2\cos\theta/2}$

$\Rightarrow \tan\phi = \dfrac{\sin\theta/2}{\cos\theta/2} \Rightarrow \tan\phi = \tan\theta/2$

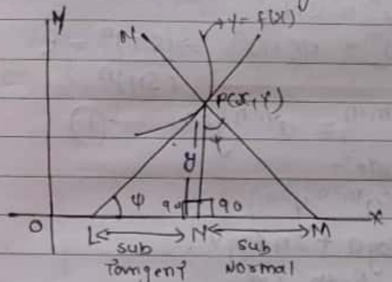
$\Rightarrow \boxed{\phi = \theta/2}$ $\quad \{\because p = r\sin\phi\}$

$\Rightarrow p = a(1-\cos\theta)\sin\theta/2$

$\Rightarrow p = a \times 2\sin^2\theta/2 \times \sin\theta/2$

$\Rightarrow p = 2a\sin^3\theta/2$ ans//

Cartesian Subtangent & Subnormal



LN = Sub Tangent

NM = Sub Normal

$\because \tan\psi = \dfrac{dy}{dx}$

$\Rightarrow$ In Δ PLN :- $\tan\psi = PN/LN$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{LN} \Rightarrow \dfrac{dx}{dy} = \dfrac{LN}{y}$

$\Rightarrow \boxed{y\dfrac{dx}{dy} = LN}$ → Sub Tangent

$\Rightarrow$ In Δ PMN :- $\tan\psi = NM/PN$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{NM}{y} \Rightarrow \boxed{y.\dfrac{dy}{dx} = NM}$ → Subnormal

Length of Tangent - BY Pythagoras Theorem

$PL = \sqrt{LN^2 + PN^2}$

$$\Rightarrow PL = \sqrt{y^2\left(\frac{dx}{dy}\right)^2 + y^2} = y\sqrt{\frac{dy}{dx}+1}$$

Length of Normal:- $PM = \sqrt{NM^2 + PN^2}$

$$\Rightarrow PM = \sqrt{y^2\left(\frac{dy}{dx}\right)^2 + y^2} = y\sqrt{\left(\frac{dy}{dx}\right)^2+1}$$

Ques 1) In this curve $x^{m+n} = a^{m-n} y^{2n}$ prove that the m^{th} power of subtangent varies as that the n^{th} power of subnormal

Sol^n To prove:- $(ST)^m \propto (SN)^n$

$$\Rightarrow (ST)^m = K(SN)^n \Rightarrow \frac{(ST)^m}{(SN)^n} = K$$

eq^n of curve $\Rightarrow x^{m+n} = a^{m-n} y^{2n}$ —(1)

Taking log Both side:-

$$\Rightarrow \log x^{(m+n)} = \log(a^{m-n} y^{2n})$$

$$\Rightarrow m+n \log x = m-n \log a + 2n \log y$$

diff w r to x Both sid

$$\Rightarrow \frac{m+n}{x} = \frac{2n}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y(m+n)}{2nx}$$

$$\left\{ \because ST = y\,dx/dy ; \quad SN = y\,dy/dx \right\}$$

$$\Rightarrow ST = y \times \frac{2nx}{y(m+n)} \Rightarrow ST = \frac{2nx}{m+n}$$

$$\Rightarrow SN = y \times \frac{y(m+n)}{2nx} \Rightarrow SN = \frac{y^2(m+n)}{2nx}$$

$$\Rightarrow \frac{(ST)^m}{(SN)^n} = \frac{\left(\frac{2nx}{m+n}\right)^m}{\left(\frac{y^2(m+n)}{2nx}\right)^n} \Rightarrow \frac{(ST)^m}{(SN)^n} = \frac{\left(\frac{2n}{m+n}\right)^m x^m}{\left(\frac{m+n}{2n}\right)^n \frac{y^{2n}}{x^n}}$$

$$\Rightarrow \frac{(ST)^m}{(SN)^n} = \left(\frac{2n}{m+n}\right)^m \times \left(\frac{2n}{m+n}\right)^n \frac{x^{m+n}}{y^{2n}}$$

$$\left\{ \text{from eq}^n \text{ (I):- } \frac{x^{m+n}}{y^{2n}} = a^{m-n} \right\}$$

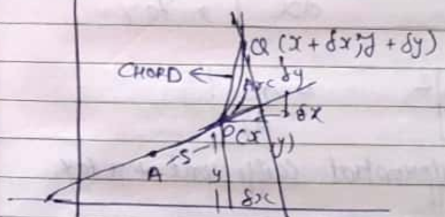
$$\Rightarrow \frac{(ST)^m}{(SN)^n} = \left(\frac{2n}{m+n}\right)^{m+n} a^{m-n} = \text{constant}$$

therefore $(ST)^m \propto (SN)^n$ Hence proved

Differential coefficient of the length of arc

$\Rightarrow$ (Cartesian form)

To find $\frac{ds}{dx}$ & $\frac{ds}{dy}$



Let A be the fixed point on the curve and $P(x,y)$ an arbitrary point on the curve such that Arc $AP = s$

Let $Q\,(x+\delta x,\ y+\delta y)$ be another point on the curve such that $AQ = s + \delta s$

when $Q \to P$, then $\delta x \to 0$

then $\lim_{\delta x \to 0} \frac{\text{Chord } PQ}{\text{Arc } PQ} = 1$ —(1)

$$\Rightarrow \lim_{\delta x\to 0} \frac{\text{Chord } PQ}{\delta x} \cdot \frac{\delta x}{\text{Arc } PQ} = 1$$

$$\Rightarrow \lim_{\delta x\to 0} \frac{\text{Chord } PQ}{\delta x} \cdot \frac{\delta x}{\delta s} = 1 \quad \{\because \text{Arc } PQ = \delta s\}$$

$$\Rightarrow \lim_{\delta x\to 0} \frac{\sqrt{(\delta x)^2+(\delta y)^2}}{\delta x} = \lim_{\delta x \to 0}\frac{\delta s}{\delta x} \quad \left\{\because PQ = \sqrt{PM^2+MQ^2} = \sqrt{(\delta x)^2+(\delta y)^2}\right.$$

$$\Rightarrow \lim_{\delta x\to 0} \sqrt{\frac{(\delta x)^2+(\delta y)^2}{(\delta x)^2}} = \frac{ds}{dx}$$

$$\Rightarrow \lim_{\delta x\to 0} \sqrt{\left[\frac{\delta x}{\delta x}\right]^2 + \left[\frac{\delta y}{\delta x}\right]^2} = \frac{ds}{dx}$$

$$\Rightarrow \boxed{\lim_{\delta x\to 0} \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{ds}{dx}}$$ → differential coefficient w. r to x

similarly

$$\Rightarrow \boxed{\lim_{\delta y\to 0} \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \frac{ds}{dy}}$$

this is the differential coefficient w. r to y.

∵ We know that $\tan \varphi = dy/dx$

then, $\sin\varphi = \dfrac{1}{\operatorname{cosec}\varphi} = \dfrac{1}{\sqrt{1+\cot^2\varphi}}$

$$\Rightarrow \sin\varphi = \frac{1}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}} \Rightarrow \sin\varphi = \frac{1}{\frac{ds}{dy}}$$

$$\Rightarrow \boxed{\sin\varphi = \frac{dy}{ds}}$$

then $\cos\varphi = \dfrac{1}{\sec\varphi} = \dfrac{1}{\sqrt{1+\tan^2\varphi}}$

$$\Rightarrow \cos\varphi = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \Rightarrow \cos\varphi = \frac{1}{\frac{ds}{dx}}$$

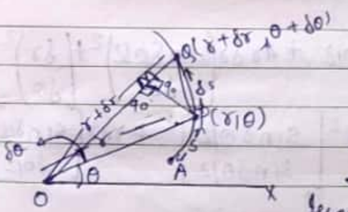
$$\Rightarrow \boxed{\cos\varphi = \frac{dx}{ds}}$$

⇒ Parametric form :- Let $x = g(t)$, $y = f(t)$

$$\Rightarrow \frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\sqrt{\left(\frac{dx}{dt}\right)^2} \Rightarrow \boxed{\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}$$ → Parametric form

⇒ Polar form :-



Let $P(r,\theta)$ & $Q(r+\delta r, \theta+\delta\theta)$ be any two neighbouring point on the curve $r = f(\theta)$

Let the length of AP measured from point A be 'S'

⇒ In ΔQMP :- $QP^2 = PM^2 + MQ^2$

⇒ $(\text{Chord } QP)^2 = PM^2 + MQ^2$

∵ In ΔMOP :- $\sin\delta\theta = \dfrac{PM}{r}$

⇒ $\boxed{r\sin\delta\theta = PM}$

⇒ $\cos\delta\theta = \dfrac{OM}{r}$ ⇒ $r\cos\delta\theta = OM$

$MQ = r + \delta r - OM \Rightarrow \boxed{MQ = r + \delta r - r\cos\delta\theta}$

Putting the value of PM & MQ

$\Rightarrow (\text{chord } QP)^2 = (r\sin\delta\theta)^2 + (r+\delta r - r\cos\delta\theta)^2$

$\Rightarrow (\text{chord } QP)^2 = r^2\sin^2\delta\theta + r^2 + (\delta r)^2 + r^2\cos^2\delta\theta + 2r\delta r - 2\delta r\, r\cos\delta\theta - 2r\cos\delta\theta\, r$

$\Rightarrow (\text{chord } QP)^2 = r^2(\sin^2\delta\theta + \cos^2\delta\theta) + r^2 + \delta r^2 + 2r\delta r - 2\delta r\, r\cos\delta\theta - 2r^2\cos\delta\theta$

$\Rightarrow (\text{chord } QP)^2 = r^2 + r^2 + \delta r^2 + 2r\delta r - 2\delta r\, r\cos\delta\theta - 2r^2\cos\delta\theta$

$\Rightarrow (\text{chord } QP)^2 = 2r^2(1-\cos\delta\theta) + 2r\delta r(1-\cos\delta\theta) + (\delta r)^2$

$\Rightarrow (\text{chord } QP)^2 = 2r^2 \times 2\sin^2\frac{\delta\theta}{2} + 2r\delta r\, 2\sin^2\frac{\delta\theta}{2} + (\delta r)^2$

$\Rightarrow (\text{chord } QP)^2 = 4r^2\sin^2\delta\theta/2 + 4r\delta r\sin^2\delta\theta/2 + (\delta r)^2$

dividing by $(\delta\theta)^2$

$$\Rightarrow \left(\frac{\text{chord } QP}{\delta\theta}\right)^2 = \frac{4r^2\sin^2\delta\theta/2}{(\delta\theta)^2} + 4r\delta r\left(\frac{\sin\delta\theta/2}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta}\right)^2$$

$$\Rightarrow \left[\frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\text{Arc } PQ}{\delta\theta}\right]^2 = r^2\left[\frac{\sin\delta\theta/2}{\sin\delta\theta/2}\right]^2 + r\delta r\left[\frac{\sin\delta\theta/2}{\delta\theta/2}\right]^2 + \left[\frac{\delta r}{\delta\theta}\right]^2$$

when $Q \to P$ then $\theta \to 0$

taking lim both sides

$$\lim_{Q\to P}\left(\frac{\text{Chord } PQ}{\text{Arc } PQ}\right) \cdot \lim_{\delta\theta\to 0}\left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$\because \text{Arc } PQ = \delta s$

$\because \lim_{\delta\theta\to 0}\frac{\delta y}{\delta\theta} = \frac{dy}{d\theta}$

$$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \Rightarrow \boxed{\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}$$

This is w.r to θ

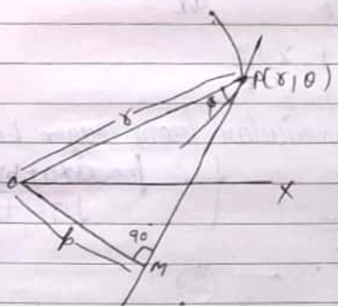
Now finding $\frac{ds}{dr}$ =

$$\Rightarrow \frac{ds}{dr} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dr} \Rightarrow \frac{ds}{dr} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\sqrt{\left(\frac{d\theta}{dr}\right)^2}$$

$$\Rightarrow \frac{ds}{dr} = \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}$$

this is w. r to 'r'

Pedal Equations = The relation b/w p & r for a given curve is called its pedal eqⁿ where is radius vector of any point on the curve and p is the length of perpendicular from the pole to the tangent of that p.



⇒ To find a pedal equation from polar eqⁿ

let $f(r,\theta) = 0$ — (1) be a given polar eqⁿ

step1 :- $p = r\sin\phi$ — (2)

step2 :- $\tan\phi = r\frac{d\theta}{dr}$, $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$ — (3)

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eliminating θ & ϕ from eq^n ①, ② & ③, then we get required pedal eq^n

NOTE:- sometime we do not get the value of ϕ from eq^n ③, then, instead of using eq^n ② & ③ we use a single relation i.e

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

⇒ To find a pedal eq^n from Cartesian eq^n

Let $f(x,y) = 0$ — ① be a given eq^n of curve

Step 1:- find eq^n of tangent:-

$$\Rightarrow (Y - y) = \frac{dy}{dx}(X - x)$$

$$\Rightarrow (Y - y) = X\frac{dy}{dx} - x\frac{dy}{dx}$$

$$\Rightarrow (Y - y) - X\frac{dy}{dx} + x\frac{dy}{dx}$$

Step 2:- Length of perpendicular from origin i.e (0,0)

$$\Rightarrow p = \frac{x\frac{dy}{dx} - y}{\sqrt{\left(1 + \frac{dy}{dx}\right)^2}} \quad \text{— ②} \qquad \left\{\because p = \frac{a\times 0 + b\times 0 + c}{\sqrt{a^2 + b^2}}\right\}$$

Step 3:- $\because r^2 = x^2 + y^2$ — ③

eliminating X and Y from eq^n ① ② & ③ then we get required pedal eq^n

Ques 1) find the pedal eq^n of ellipse $\frac{l}{r} = 1 + e\cos\theta$

Sol^n eq^n of curve $= \frac{l}{r} = 1 + e\cos\theta$ — ①

diff. w.r. to 'θ', we get

$$\Rightarrow \frac{+l}{r^2}\frac{dr}{d\theta} = +e\sin\theta$$

$$\Rightarrow \frac{1}{r^2}\frac{dr}{d\theta} = \frac{e\sin\theta}{l} \quad \text{— ②}$$

$$\left\{\because \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{e\sin\theta}{l}\right)^2 \Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2\sin^2\theta}{l^2}$$

$$\{\because \sin^2\theta = 1 - \cos^2\theta\} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2(1 - \cos^2\theta)}{l^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2 - e^2\cos^2\theta}{l^2} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{(e\cos\theta)^2}{l^2}$$

$$\left\{\because \text{from eq}^n \text{ ① } e\cos\theta = \frac{l}{r} - 1 \Rightarrow \frac{l - r}{r} = e\cos\theta\right\}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{l^2}\left(\frac{l - r}{r}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{l^2}\left[\frac{l^2 + r^2 - 2lr}{r^2}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \left[\frac{\cancel{l^2}}{\cancel{l^2}r^2} + \frac{\cancel{r^2}}{l^2\cancel{r^2}} - \frac{2lr}{l^2r^2}\right]$$

$\Rightarrow \frac{1}{p^2} = \cancel{\frac{1}{r^2}} + \frac{e^2}{l^2} - \cancel{\frac{1}{r^2}} - \frac{1}{l^2} + \frac{2}{lr}$

$\Rightarrow \frac{1}{p^2} = \frac{e^2}{l^2} - \frac{1}{l^2} + \frac{2}{lr} \times \frac{l}{l}$ [M & D by l in N & D]

$\Rightarrow \frac{1}{p^2} = \frac{1}{l^2}\left[e - 1 + \frac{2l}{r}\right]$ ans//

Ques 2) Find the pedal eqⁿ of parabola $y^2 = 4a(x+a)$

Sol eqⁿ of curve:- $y^2 = 4a(x+a)$ —(1)

diff. w. r. to x, we get

$\Rightarrow \cancel{2}y\frac{dy}{dx} = {}^{2}\cancel{4}a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

$\left\{ \because p = \frac{x\frac{dy}{dx} - y}{\sqrt{1+(dy/dx)^2}} \right\}$

$\Rightarrow p = \frac{\frac{2ax}{y} - y}{\sqrt{1+\left(\frac{2a}{y}\right)^2}} \Rightarrow p = \frac{\frac{2ax - y^2}{\cancel{y}}}{\frac{\sqrt{y^2+4a^2}}{\cancel{y}}}$

$\Rightarrow p = \frac{2ax - y^2}{\sqrt{y^2+4a^2}} \Rightarrow p = \frac{2ax - 4a(x+a)}{\sqrt{4a(x+a)+4a^2}}$ [From eqⁿ (1)]

$\Rightarrow p = \frac{2ax - 4ax - 4a^2}{\sqrt{4ax+4a^2+4a^2}} = \frac{-2ax - 4a^2}{\sqrt{4ax+8a^2}}$

$\Rightarrow p = \frac{-2a(x+2a)}{\sqrt{4a(x+2a)}}$

$\Rightarrow p = \frac{-\cancel{2}(\sqrt{a})^2(\sqrt{x+2a})^2}{\cancel{2}\sqrt{a}(\sqrt{x+2a})} \Rightarrow p = -\sqrt{a}(\sqrt{x+2a})$

$\Rightarrow p = -\sqrt{a(x+2a)}$ ans —(2)

$\Rightarrow$ squaring both side we get

$\Rightarrow \frac{p^2}{a} = x+2a$

we know that, $r^2 = x^2+y^2$

$\Rightarrow r^2 = x^2 + 4a(x+a) = x^2+4ax+4a^2$

$\Rightarrow r^2 = x^2+4ax+(2a)^2$

$\Rightarrow r^2 = (x+2a)^2 \Rightarrow r^2 = \left(\frac{p^2}{a}\right)^2$

$\Rightarrow r = \frac{p^2}{a} \Rightarrow \boxed{ar = p^2}$ ans//

Ques 3) Show that the pedal eqⁿ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$

Sol eqⁿ of curve:- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ —(1)

∵ The parametric eqⁿ of given curve is:- $x = a\cos t$, $y = b\sin t$

diff w. r to 't'; we get:-

$\Rightarrow \frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = b\cos t$

$\Rightarrow \frac{dy}{dx} = -\frac{b\cos t}{a\sin t} \Rightarrow \frac{dy}{dx} = -\frac{b}{a}\cot t$

$$\left\{\because\ p=\frac{x\frac{dy}{dx}-y}{\sqrt{1+(dy/dx)^2}}\right\} \Rightarrow p=\frac{a\cos t\left[\frac{-b}{a}\cot t\right]-b\sin t}{\sqrt{1+\left(\frac{-b\cot t}{a}\right)^2}}$$

$$\Rightarrow p=\frac{-ab\cos t\left[\frac{-b\cos t}{a\sin t}\right]-b\sin t}{\sqrt{1+\frac{b^2\cos^2 t}{a^2\sin^2 t}}}$$

$$\Rightarrow p=\frac{\frac{-ab\cos^2 t-ab\sin^2 t}{a\sin t}}{\frac{\sqrt{a^2\sin^2 t+b^2\cos^2 t}}{a\sin t}}$$

$$\Rightarrow p=\frac{-ab\cos^2 t-ab\sin^2 t}{\sqrt{a^2\sin^2 t+b^2\cos^2 t}} \Rightarrow p=\frac{-ab}{\sqrt{a^2\sin^2 t+b^2\cos^2 t}}$$

squaring and reversing both side

$$\Rightarrow \frac{1}{p^2}=\frac{a^2\sin^2 t+b^2\cos^2 t}{a^2b^2} \quad \text{--- (2)}$$

$$\{\because\ r^2=x^2+y^2=a^2\sin^2 t+b^2\cos^2 t\} \quad \text{--- (3)}$$

$$\Rightarrow \frac{1}{p^2}=\frac{a^2(1-\cos^2 t)+b^2(1-\sin^2 t)}{a^2b^2}$$

$$\Rightarrow \frac{1}{p^2}=\frac{a^2-a^2\cos^2 t+b^2-b^2\sin^2 t}{a^2b^2}$$

$$\Rightarrow \frac{1}{p^2}=\frac{a^2+b^2-(a^2\cos^2 t+b^2\sin^2 t)}{a^2b^2}$$

$$\Rightarrow \frac{1}{p^2}=\frac{a^2+b^2-r^2}{a^2b^2} \Rightarrow \frac{1}{p^2}=\frac{a^2}{a^2b^2}+\frac{b^2}{a^2b^2}-\frac{r^2}{a^2b^2}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}$$

Hence Proved

Radius Of Curvature

The bending of a curve at a given point on it is called curvature

$\Rightarrow$ In cartesian form: $\rho=\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$\Rightarrow$ In parametric form $=\dfrac{[(x')^2+(y')^2]^{3/2}}{x'y''-y'x''}$

$\{\because\ x'=dx/dt,\ y'=dy/dt\}$

Ques) find the radius of curvature of $x^2+y^2=a^2$

Sol^n eq^n:- $x^2+y^2=a^2$

diff. w.r. to 'x', we get

$$\Rightarrow 2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$$

again diff. w.r. to x, we get

$$\Rightarrow \frac{d^2y}{dx^2}=-\left[\frac{y-x\frac{dy}{dx}}{y^2}\right] \Rightarrow \frac{d^2y}{dx^2}=-\left[\frac{y-x\cdot\left[\frac{-x}{y}\right]}{y^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{y + \frac{x^2}{y}}{y^2}\right] \Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{y^2+x^2}{y^3}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{a^2}{y^3}\right]$$

$$\left\{\because \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}\right\}$$

$$\Rightarrow \rho = \frac{\left[1+\left(\frac{-x}{y}\right)^2\right]^{3/2}}{-(a^2/y^3)} \Rightarrow \rho = \frac{\left[1+\frac{x^2}{y^2}\right]^{3/2} y^3}{-a^2}$$

$$\Rightarrow \rho = \frac{-y^3}{a^2}\left[\frac{y^2+x^2}{y^2}\right]^{3/2} \Rightarrow \rho = \frac{y^3}{-a^2}\left[\frac{a^2}{y^2}\right]^{3/2}$$

$$\Rightarrow \rho = \frac{y^3}{-a^2} \times \frac{a^3}{y^3} \Rightarrow \boxed{\rho = a} \text{ ans//}$$

Ques 2) If $x = a\cos t, y = a\sin t$ find radius of curvature

Sol^n $x' = -a\sin t$, $y' = a\cos t$

$x'' = -a\cos t$, $y'' = -a\sin t$

$$\left\{\because \rho = \frac{[(x')^2 + (y')^2]^{3/2}}{x'y'' - y'x''}\right\}$$

$$\Rightarrow \rho = \frac{[(-a\sin t)^2 + (a\cos t)^2]^{3/2}}{(-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t)}$$

$$\Rightarrow \rho = \frac{a^3[\sin^2 t + \cos^2 t]^{3/2}}{a^2(\sin^2 t + \cos^2 t)}$$

$$\Rightarrow \boxed{\rho = a} \text{ ans//}$$

Ques 3) find the radius of curvature of $\sqrt{x} + \sqrt{y} = 1$ at $(1/4, 1/4)$

Sol^n eq^n = $\sqrt{x} + \sqrt{y} = 1$

diff. w. r. to 'x', we get

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}} \quad \text{—} (1)$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(\frac{1}{4},\frac{1}{4})} = -\frac{\sqrt{1/4}}{\sqrt{1/4}} \Rightarrow \left[\frac{dy}{dx}\right]_{(1/4,1/4)} = -1$$

again diff. w. r. to 'x', we get

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \frac{1}{2\sqrt{x}}\sqrt{y}}{x}\right]$$

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{(1/4,1/4)} = -\left[\frac{\sqrt{1/4}\,\frac{1}{2\sqrt{1/4}}(-1) - \frac{1}{2\sqrt{1/4}}\sqrt{1/4}}{1/4}\right]$$

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{(1/4,1/4)} = -\left[\frac{-1/2 - 1/2}{1/4}\right] = -(-1)\times 4$$

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{(\frac{1}{4},\frac{1}{4})} = 4$$

$$\Rightarrow \rho = \frac{[1+(-1)^2]^{3/2}}{4} = \frac{(2)^{3/2}}{(2)^2} = (2)^{-1/2}$$

$$\Rightarrow \boxed{\rho = \frac{1}{\sqrt{2}}}$$

Ques4) In curve $y = ae^{x/a}$ prove that $\rho = a\sec^2\theta\,\mathrm{cosec}\,\theta$ where $\theta = \tan^{-1}(y/a)$

Sol^n eq^n :- $y = ae^{x/a}$

diff. w. r. to 'x', we get

$\Rightarrow \frac{dy}{dx} = a \times \frac{1}{a} e^{x/a} \Rightarrow \frac{dy}{dx} = e^{x/a}$

again diff. w. r. to x.

$\Rightarrow \frac{d^2y}{dx^2} = \frac{e^{x/a}}{a}$

$\left\{ \because \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} \right\} \Rightarrow \rho = \frac{\left[1 + (e^{x/a})^2\right]^{3/2}}{\frac{e^{x/a}}{a}}$

$\Rightarrow \rho = \frac{a\left[1 + (e^{x/a})^2\right]^{3/2}}{e^{x/a}}$

$\left\{ \because \frac{y}{a} = e^{x/a} \quad \& \quad \theta = \tan^{-1}(y/a) \Rightarrow \tan\theta = y/a \Rightarrow \boxed{\tan\theta = e^{x/a}} \right\}$

$\Rightarrow \rho = \frac{a\left[1 + \tan^2\theta\right]^{3/2}}{\tan\theta} = \frac{a(\sec^2\theta)^{3/2}}{\frac{\sin\theta}{\cos\theta}}$

$\Rightarrow \rho = \frac{a\sec^3\theta\,\mathrm{cosec}\,\theta}{\sec\theta} \Rightarrow \boxed{\rho = a\sec^2\theta\,\mathrm{cosec}\,\theta}$ ans//

Ques5) find the radius of curvature of following case $y^2 = \frac{4a^2(2a - x)}{x}$ at its vertex

Sol^n $xy^2 = 4a^2(2a - x)$ — (1)

NOTE= Here the power of Y is even it means jitna curve 'x axis' k upar hoga] utna hi niche bhi hoga

It is symmetric to x axis it means it cuts the x axis at a point where Y coordinate is 0, so putting $Y=0$ in eq^n (1)

$\Rightarrow 0 = 8a^3 - 4a^2x \Rightarrow x = \frac{{}^2 8a^3}{4a^2}$

$\Rightarrow \boxed{x = 2a}$, so required points are $(2a, 0)$

Now, at this point we need to find radius of curvature.

$\Rightarrow xy^2 = 8a^3 - 4a^2x \Rightarrow xy^2 + 4a^2x = 8a^3$

$\Rightarrow x(y^2 + 4a^2) = 8a^3 \Rightarrow x = \frac{8a^3}{y^2 + 4a^2}$

$\Rightarrow \frac{dx}{dy} = \frac{8a^3(2y)}{(y^2 + 4a^2)^2} \Rightarrow \left[\frac{dx}{dy}\right]_{(2a,0)} = 0$

$\Rightarrow \frac{dx}{dy} = \frac{16a^3 y}{(y^2 + 4a^2)^2}$ again diff. w. r. to y

$\Rightarrow \frac{d^2x}{dy^2} = 16a^3\left[\frac{(y^2 + 4a^2)^2 \cdot 1 - y \cdot 2(y^2 + 4a^2) \cdot 2y}{(y^2 + 4a^2)^4}\right]$

$\Rightarrow \left[\frac{d^2x}{dy^2}\right]_{(2a,0)} = 16a^3\left[\frac{(4a^2)^2 - 0}{(4a^2)^4}\right] \Rightarrow \left[\frac{d^2x}{dy^2}\right]_{(2a,0)} = \frac{16a^3}{16a^4}$

$\Rightarrow \left[\frac{d^2x}{dy^2}\right]_{(2a,0)} = \frac{1}{a}$

$\Rightarrow \rho = a\left[1 + (0)^2\right]^{3/2} \Rightarrow \boxed{\rho = a}$ ans//

Ques6) find radius of curvature at point (x, y) of $x^{2/3} + y^{2/3} = a^{2/3}$ or show that

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R at $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$

Sol^n $\Rightarrow x^{2/3}+y^{2/3}=a^{2/3}$

diff. w.r. to x, we get

$$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} \Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

again diff. w.r. to x, we get

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\left[y^{1/3}\frac{1}{3}x^{-2/3} - x^{1/3}\frac{1}{3}y^{-2/3}\frac{dy}{dx}\right]}{(x^{1/3})^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{3x^{2/3}}\left[\frac{y^{1/3}}{x^{2/3}} - \frac{\cancel{x^{1/3}}}{y^{2/3}}\right]\left[\frac{-y^{1/3}}{\cancel{x^{1/3}}}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{3x^{2/3}}\left[\frac{y^{1/3}}{x^{2/3}} + \frac{y^{1/3}}{y^{2/3}}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{3x^{2/3}}\left[\frac{y^{1/3}}{x^{2/3}} + \frac{1}{y^{1/3}}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{3x^{2/3}}\left[\frac{y^{2/3}+x^{2/3}}{x^{2/3}\cdot y^{1/3}}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{3x^{2/3}}\left[\frac{a^{2/3}}{x^{2/3}y^{1/3}}\right] \Rightarrow \frac{d^2y}{dx^2} = \frac{-a^{2/3}}{3x^{4/3}y^{1/3}}$$

$$\Rightarrow \rho = \frac{\left[1+\frac{y^{2/3}}{x^{2/3}}\right]^{3/2}}{\frac{-a^{2/3}}{3x^{4/3}y^{1/3}}} \Rightarrow \rho = \left[\frac{x^{2/3}+y^{2/3}}{x^{2/3}}\right]^{3/2}\frac{3x^{4/3}y^{1/3}}{a^{2/3}}$$

$$\Rightarrow \rho = \frac{[a^{2/3}]^{3/2}}{[x^{2/3}]^{3/2}}\frac{3x^{4/3}y^{1/3}}{a^{2/3}} \Rightarrow \rho = \frac{3x^{4/3}y^{1/3}}{a^{2/3}}\frac{a}{x}$$

$$\Rightarrow \rho = \frac{3x^{4/3}y^{1/3}x^{-1}}{a^{2/3}a^{-1}} \Rightarrow \rho = \frac{3x^{1/3}y^{1/3}}{a^{-1/3}}$$

$$\Rightarrow \rho = 3x^{1/3}y^{1/3}a^{1/3} \Rightarrow \rho = 3(axy)^{1/3}$$

Ques7) At any point p of parabola $y^2 = 4ax$

PT (a) $\rho = \frac{2}{\sqrt{a}}(x+a)^{3/2} = \frac{2(SP)^{3/2}}{\sqrt{a}}$ where S = focus

(b) $(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3} = (l)^{-2/3}$ where ρ_1 & ρ_2 are radii of curvature at end of any focus chord & l is length of semi latus rectum

Sol^n (a) parametric form of parabola :- $x = at^2$, $y = 2at$

$\Rightarrow x' = 2at$, $x'' = 2a$, $y' = 2a$, $y'' = 0$

$$\left\{\because \rho = \frac{[(x')^2+(y')^2]^{3/2}}{x'y''-y'x''}\right\}$$

$$\Rightarrow \rho = \frac{[(2at)^2+(2a)^2]^{3/2}}{2at\times 0 - 2a\times 2a} \Rightarrow \rho = \frac{[4a^2t^2+4a^2]^{3/2}}{-4a^2}$$

$$\Rightarrow \rho = \frac{(4a^2)^{3/2}[t^2+1]^{3/2}}{4a^2} \Rightarrow \rho = \frac{8a^3[t^2+1]^{3/2}}{4a^2}$$

$[\because x = at^2, \Rightarrow t^2 = x/a]$

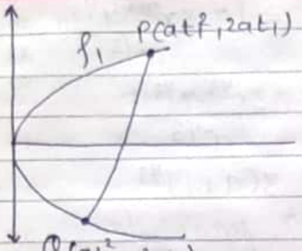
$$\Rightarrow \rho = 2a\left[\frac{x}{a}+1\right]^{3/2} \Rightarrow \rho = 2a\left[\frac{x+a}{a}\right]^{3/2}$$

$$\Rightarrow \rho = \frac{2a[x+a]^{3/2}}{a\sqrt{a}} \Rightarrow \rho = \frac{2}{\sqrt{a}}(x+a)^{3/2}$$

By focal property :- $sp = x+a$

$$\Rightarrow \rho = \frac{2}{\sqrt{a}}(sp)^{3/2}$$

(b)



$\Rightarrow t_1 \cdot t_2 = -1$

$\Rightarrow t_2 = -1/t_1$

$\because \rho = 2a[t^2+1]^{3/2}$

$\Rightarrow$ Radius of Curvature at $P(at_1^2, 2at_1)$

$\rho_1 = 2a[t_1^2+1]^{3/2}$

$\Rightarrow$ Radius of Curvature at $Q(at_2^2, 2at_2)$

$\rho_2 = 2a[t_2^2+1]^{3/2}$

$L.H.S = (\rho_1)^{-2/3} + (\rho_2)^{-2/3}$

$\Rightarrow [(2a[t_1^2+1])^{3/2}]^{-2/3} + [(2a[t_2^2+1])^{3/2}]^{-2/3}$

$\Rightarrow (2a)^{-2/3} \dfrac{1}{1+t_1^2} + (2a)^{-2/3} \dfrac{1}{t_2^2+1}$

$\Rightarrow (2a)^{-2/3}\left[\dfrac{1}{1+t_1^2} + \dfrac{1}{1+1/t_1^2}\right]$

$\Rightarrow (2a)^{-2/3}\left[\cancel{\dfrac{1}{1+t_1^2} + \dfrac{t_1^2}{1+t_1^2}}\right]$

$\Rightarrow \{\because$ Latus rectum of parabola $= 4a$
Semi latus rectum $= l \}$

$\Rightarrow (2a)^{-2/3} = (l)^{-2/3}$ ans//

Ques 8) At any point P of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Prove that :-

(a) $\rho = \dfrac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$

(b) $\rho = \dfrac{a^2b^2}{p^3}$ where P is the length of ⊥ drawn from centre on the Tangent at P

(c) $\rho = \dfrac{(CD)^3}{ab}$ where CP & CD is pair of Conjugate diameters

(d) $(ab)^{2/3}(\rho_1^{2/3} + \rho_2^{2/3}) = a^2 + b^2$, where ρ_1 & ρ_2 are Radii of curvature at the end of extremities of Conjugate diameter

Sol: parametric form :- $x = a\cos t$, $y = b\sin t$

diff. w. r. to t, we get

$\Rightarrow x' = -a\sin t$, $y' = b\cos t$

$x'' = -a\cos t$, $y'' = -b\sin t$

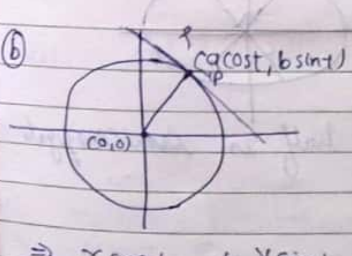
$\left\{\because \rho = \dfrac{[(x')^2 + (y')^2]^{3/2}}{x'y'' - y'x''}\right\}$

$\Rightarrow \rho = \dfrac{[(-a\sin t)^2 + (b\cos t)^2]^{3/2}}{(-a\sin t)(-b\sin t) - (b\cos t)(-a\cos t)}$

$\Rightarrow \rho = \dfrac{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}{ab\sin^2 t + ab\cos^2 t} \Rightarrow \rho = \dfrac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab[\sin^2 t + \cos^2 t]}$

$\Rightarrow \boxed{\rho = \dfrac{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}{ab}}$ ans of part (a)

(b)



eqⁿ of Tangent at P -

$\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$

$\Rightarrow \dfrac{x}{a^2}(a\cos t) + \dfrac{y}{b^2}b\sin t = 1$

$\Rightarrow \dfrac{x\cos t}{a} + \dfrac{y\sin t}{b} = 1$

$\left\{\because \text{distance formula} = \left|\dfrac{ax+by+c}{\sqrt{a^2+b^2}}\right|\right\}$

putting $(0,0)$ in place of x and Y. we get

$$\Rightarrow p = \frac{|0+0-1|}{\sqrt{\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2}}} \Rightarrow \frac{ab}{\sqrt{b^2\cos^2 t + a^2\sin^2 t}}$$

doing reciprocal :-

$$\Rightarrow \frac{1}{p} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{1/2}}{ab}$$

taking cube both side :-

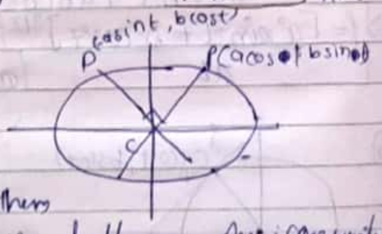
$$\Rightarrow \frac{1}{p^3} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{(ab)^3}$$

multiplying by a^2b^2 both side

$$\Rightarrow \frac{a^2b^2}{p^3} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{(ab)^3}(a^2b^2)$$

$$\Rightarrow \frac{a^2b^2}{p^3} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{(ab)} \Rightarrow \boxed{\frac{a^2b^2}{p^3} = \rho}$$ HP.

(c) PT $\Rightarrow \rho = \frac{(CD)^3}{ab}$



conjugate diameter are those which have angle of 90° b/w them like CP & CD and there half is semi conjugate diameter (angle is same)

$$CD = \sqrt{(x_1-x_2)^2 + (Y_1-Y_2)^2}$$

$$\Rightarrow CD = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

Taking cube both side

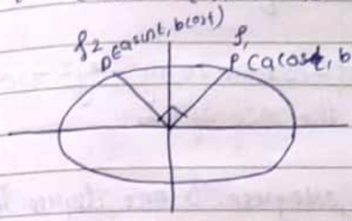
$$\Rightarrow (CD)^3 = (a^2 b^2 \sin^2 t + b^2\cos^2 t)^{3/2}$$

dividing by ab both side

$$\Rightarrow \frac{(CD)^3}{ab} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$$

$$\Rightarrow \frac{(CD)^3}{ab} = \rho$$ Hence Proved

(d) PT $= (ab)^{2/3}(\rho_1^{2/3} + \rho_2^{2/3}) = a^2+b^2$



Radius of curvature at $P(a\cos t, b\sin t)$

$$\Rightarrow \rho_1 = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$$

Radius of curvature at $D(-a\sin t, b\cos t)$

$$\Rightarrow \rho_2 = \frac{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}}{ab}$$ [ρ_1 mai sin ki jagah cos or cos ki jagah sin rakha hai]

LHS :- $(ab)^{2/3}(\rho_1^{2/3} + \rho_2^{2/3}) = (ab)^{2/3}\left[\frac{(a^2\sin^2 t + b^2\cos^2 t)}{(ab)^{2/3}} + \frac{a^2\cos^2 t + b^2\sin^2 t}{(ab)^{2/3}}\right]$

$\Rightarrow a^2\sin^2 t + b^2\cos^2 t + a^2\cos^2 t + b^2\sin^2 t$

$\Rightarrow a^2(\sin^2 + \cos^2 t) + b^2(\cos^2 t + \sin^2 t)$

$\Rightarrow a^2 + b^2$ Hence Proved

Asymptotes

It is a tangent at infinite. The Highest degree of asymptotes eqn determines how many asymptotes are there. Let us suppose the highest degree is 2 then we need to find 2 straight line eqns.

Ques1) $y^3 - 3xy^2 - x^2y + 3x^3 - 3x^2 + 10xy - 3y^2 - 10x - 10y + 7 = 0$ find the asymptotes

Soln Highest degree = 3

separate 3 degree, 2 degree & one degree terms

put $x = 1$, $y = m$

$\phi_3(m) = m^3 - 3m^2 - m + 3$

$\phi_2(m) = -3 + 10m - 3m^2$

$\phi_1(m) = -10 - 10m$

Put the highest degree $= 0$ i.e. $\phi_3(m) = 0$

$\Rightarrow m^3 - 3m^2 - m + 3 = 0$

$\Rightarrow m^2(m-3) - 1(m-3) = 0$

$\Rightarrow (m^2 - 1)(m-3) = 0$

$\Rightarrow (m-1)(m+1)(m-3) = 0$

$\Rightarrow m_1 = 1, \; m_2 = -1, \; m_3 = 3$

when the value of m are diff then the formula for c is :-

$$\Rightarrow c = -\frac{\phi_2(m)}{\phi_3'(m)}$$

$\Rightarrow \phi_3'(m) = 3m^2 - 6m - 1$

$$\Rightarrow c = -\left[\frac{-3 + 10m - 3m^2}{3m^2 - 6m - 1}\right]$$

Put $m_1 = 1$, then we'll get c_1 :-

$$\Rightarrow c_1 = -\left[\frac{-3 + 10\times 1 - 3\times 1^2}{3\times 1^2 - 6\times 1 - 1}\right] = -\left[\frac{10 - 6}{-4}\right]$$

$\Rightarrow c_1 = \frac{-4}{-4} \Rightarrow \boxed{c_1 = 1}$

Put $m_2 = -1$, then we'll get c_2 :-

$$\Rightarrow c_2 = -\left[\frac{-3 - 10 - 3}{3 + 6 - 1}\right] = -\left[\frac{-16}{8}\right]$$

$\Rightarrow \boxed{c_2 = 2}$

Put $m_3 = 3$, then we'll get c_3 :-

$$\Rightarrow c_3 = -\left[\frac{-3 + 30 - 27}{27 - 18 - 1}\right] \Rightarrow \boxed{c_3 = 0}$$

$\Rightarrow y = m_1x + c_1 \Rightarrow y = x + 1 \Rightarrow \boxed{y - x - 1}$

$\Rightarrow y = m_2x + c_2 \Rightarrow y = -x + 2 \Rightarrow \boxed{y + x - 2}$

$\Rightarrow y = m_3x + c_3 \Rightarrow y = 3x + 0 \Rightarrow \boxed{y - 3x = 0}$

Ques2) $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0$

Soln put $x = 1$, $y = m$

$\phi_3(m) = 1 - 5m + 8m^2 - 4m^3$

$\phi_2(m) = 1 - 3m + 2m^2$

$\phi_1(m) = 0$

Put $\phi_3(m) = 0$, then we get

$\Rightarrow 1 - 5m + 8m^2 - 4m^3 = 0$

$\Rightarrow 4m^3 - 8m^2 + 5m - 1 = 0$

If we put $m = 1$ then it will be 0 i.e. $(m-1)$ is a factor of eqn

$\Rightarrow 4m^2(m-1) - 4m(m-1) + 1(m-1) = 0$

$\Rightarrow (m-1)[4m^2 - 4m + 1] = 0$

$\Rightarrow (m-1)(2m-1)^2 = 0$

$\Rightarrow m = 1, 1/2, 1/2$

for $m_1 = 1$, $\left\{ c = -\dfrac{\phi_2(m)}{\phi_3'(m)} \right\}$

$\Rightarrow \phi_3'(m) = -12m^2 + 16m - 5$

$\Rightarrow c_1 = -\left[\dfrac{1 - 3m + 2m^2}{-12m^2 + 16m - 5}\right] = -\left[\dfrac{1-3+2}{-12+16-5}\right]$

$\Rightarrow \boxed{c_1 = 0}$

$\Rightarrow y = m_1 x + c_1 \Rightarrow y = x + 0 \Rightarrow \boxed{y - x = 0}$

When the value of m is same, formula for c will be :- $\dfrac{c^2}{2!}\phi_3''(m) + \dfrac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$

$\Rightarrow \phi_3''(m) = -24m + 16$

$\Rightarrow \phi_2'(m) = 4m - 3$

$\Rightarrow \dfrac{c^2}{2!}[-24m + 16] + c[4m - 3] + 0 = 0$

Putting $m = 1/2$

$\Rightarrow \dfrac{c^2}{2}[16 - 12] + c[2 - 3] = 0$

$\Rightarrow 2c^2 - c = 0 \Rightarrow c(2c - 1) = 0$

$\Rightarrow c_2 = 0, c_3 = 1/2$

$\Rightarrow y = m_2 x + c_2 \Rightarrow y = 1/2\, x + 0 \Rightarrow \boxed{2y - x = 0}$

$\Rightarrow y = m_3 x + c_3 \Rightarrow y = 1/2\, x + 1/2 \Rightarrow \boxed{2y - x - 1 = 0}$

Ques 3) $(x+y)^2(x+2y+2) = x + 9y - 2$

Sol^n $(x+y)^2(x+2y+2) - x - 9y + 2 = 0$

$\Rightarrow (x+y)^2(x+2y) + 2(x+y)^2 - x - 9y + 2 = 0$

Put $x = 1$, $y = m$

$\Rightarrow \phi_3(m) = (1+m)^2(1+2m)$

$\phi_2(m) = 2(1+m)^2$

$\phi_1(m) = -1 - 9m$

Putting $\phi_3(m) = 0$

$\Rightarrow (1+m)^2(1+2m) = 0$

$\Rightarrow m = -1, -1, -1/2$

for $m_1 = -1/2$, $c = -\left[\dfrac{\phi_2(m)}{\phi_3'm}\right]$

$\Rightarrow \phi_3'(m) = (1+m)^2 \cdot 2 + (1+2m) \cdot 2(1+m)$

$\Rightarrow \phi_3'(m) = 2(1+m)[(1+m) + (1+2m)]$

$\Rightarrow \phi_3'(m) = 2(1+m)[2 + 3m]$

$\Rightarrow c_1 = -\left[\dfrac{\cancel{2}(1+m)^{\cancel{2}}}{\cancel{2}\cancel{(1+m)}(2+3m)}\right]$

$\Rightarrow c_1 = -\left[\dfrac{(1+m)}{2+3m}\right]$

$\Rightarrow c_1 = -\left[\dfrac{(1 - 1/2)}{2 - 3 \times 1/2}\right] \Rightarrow c_1 = -\left[\dfrac{\cancel{1/2}}{\cancel{1/2}}\right]$

$\Rightarrow \boxed{c_1 = -1}$

$y = m_1 x + c_1 \Rightarrow y = -\dfrac{1}{2}x - 1 \Rightarrow \boxed{2y + x + 1 = 0}$

for $m_2 = m_3 = -1$, $\dfrac{c^2}{2!}\phi_3''(m) + \dfrac{c}{1!}\phi_2'(m) + \phi_1(m)$

$\Rightarrow \phi_3''(m) = 2[2+3m + (1+m)3]$

$\Rightarrow \phi_3''(m) = 2[2+3m+3+3m]$

$\Rightarrow \phi_3''(m) = 2[5+6m] \Rightarrow \boxed{\phi_3''(m) = 10+12m}$

$\Rightarrow \boxed{\phi_2'(m) = 4(1+m)}$

$\Rightarrow \frac{c^2}{2!}(10+12m) + \frac{c}{1!}4(1+m) + (-1-9m) = 0$

$\Rightarrow \frac{c^2}{2}(10-12) + c\cdot 0 - 1 + 9 = 0$

$\Rightarrow -c^2 + 8 = 0 \quad \Rightarrow c^2 = 8 \quad \Rightarrow c = 2\sqrt{2}$

$\Rightarrow c_2 = 2\sqrt{2}$	, $c_3 = -2\sqrt{2}$
$m_2 = -1$	, $m_3 = -1$
$y = m_2x + c_2$	$y = m_3x + c_3$
$y = -x + 2\sqrt{2}$	$y = -x - 2\sqrt{2}$
$\boxed{y + x - 2\sqrt{2} = 0}$	$\boxed{y + x + 2\sqrt{2} = 0}$

Intersection of the Curve & its Asymptotes -

Ques) Show that the four asymptote of the curve $(x^2-y^2)(y^2-4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$ cut the curve again in eight points which lie on the circle $x^2+y^2=1$

Sol^n Put $x = 1, y = m$

$\Rightarrow \phi_4(m) = (1-m^2)(m^2-4)$

$\phi_3(m) = 6 - 5m - 3m^2 + 2m^3$

$\phi_2(m) = -1 + 3m$

$\phi_1(m) = 0$

Let $\phi_4(m) = 0$

$\Rightarrow (1-m^2)(m^2-4) = 0$

$\Rightarrow m = 1, -1, 2, -2$

$\phi_4'(m) = -2m(m^2-4) + 2m(1-m^2)$

$\Rightarrow c = -\left[\frac{\phi_3 m}{\phi_4' (m)}\right] \Rightarrow \cancel{c = -\left[\frac{-1+3m}{10m-4m^3}\right]}$

$\left\{\begin{array}{l} \because \phi_4'(m) = -2m(m^2-4) + 2m(1-m^2) \\ \quad = 2m[-m^2 + 4 + 1 - m^2] \\ \quad = 2m[5 - 2m^2] \\ \phi_4' m = 10m - 4m^3 \end{array}\right\}$

$\Rightarrow c = -\left[\frac{6 - 5m - 3m^2 + 2m^3}{10m - 4m^3}\right]$

(1) for $m_1 = 1$

$\Rightarrow c_1 = -\left[\frac{6-5-3+2}{10-4}\right] \Rightarrow \boxed{c_1 = 0}$

$y = m_1x + c_1 \Rightarrow y = x + 0 \Rightarrow \boxed{y - x = 0}$

(2) for $m_2 = -1$

$\Rightarrow c_2 = -\left[\frac{6+5-3-2}{-10+4}\right] \Rightarrow \boxed{c_2 = 1}$

$y = m_2x + c_2 \Rightarrow y = -x + 1 \Rightarrow \boxed{y + x - 1 = 0}$

(3) for $m_3 = 2$

$\Rightarrow c_3 = -\left[\frac{6-10-12+16}{20-32}\right] \Rightarrow \boxed{c_3 = 0}$

$y = m_3x + c_3 \Rightarrow y = 2x + 0 \Rightarrow \boxed{y - 2x = 0}$

(4) for $m_4 = -2$

$c_4 = -\left[\frac{6+10-12-16}{-20+32}\right] \Rightarrow \boxed{c_4 = 1}$

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$Y = m_4 x + c_4 \Rightarrow Y = -2x+1 \Rightarrow \boxed{Y+2x-1=0}$

Given eqⁿ of curve $= (x^2-y^2)(y^2-4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$ — (1)

Multiply all the 4 asymptote

$\Rightarrow (Y-x)(Y+x-1)(Y-2x)(Y+2x-1) = 0$ — (2)

Subtracting both the eqⁿ

$\Rightarrow (x^2-y^2)(y^2-4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 - (Y-x)(Y+x-1)(Y-2x)(Y+2x-1) = 0$

$x^2+y^2 = 1$ Hence Proved

which circle of unit radius and int lamp out at $n(n-2)$ points i.e $4(4-2)$ points $= 8$ points [n = higher degree]

Ques) Show that the Asymptote of the following cubic curve cut the curve again in three points which lie on straight line $x-y+1=0$, $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$

Sol^n $x^3-2y^3+2x^2y-xy^2+xy-y^2+1=0$

Put $x=1$, $Y=m$

$\phi_3(m) = 1-2m^3+2m-m^2$

$\phi_2(m) = m-m^2$

$\phi_1(m) = 0$

Putting $\phi_3(m) = 0$

$\Rightarrow 1-2m^3+2m-m^2 = 0$

$\Rightarrow 2m^3+m^2-2m-1 = 0$

$\Rightarrow m^2(2m+1)-1(2m+1) = 0$

$\Rightarrow (m^2-1)(2m+1) = 0 \Rightarrow m = 1, -1, -1/2$

$\Rightarrow \phi_3'(m) = -6m^2+2-2m$

$\Rightarrow c = -\left[\frac{\phi_2(m)}{\phi_3'(m)}\right] \Rightarrow c = -\left[\frac{m-m^2}{-6m^2+2-2m}\right]$

(1) for $m_1 = 1$

$\Rightarrow c_1 = -\left[\frac{1-1^2}{-6+2-2}\right] \Rightarrow \boxed{c_1 = 0}$

$Y = m_1x + c_1 \Rightarrow Y = x \Rightarrow \boxed{Y-x=0}$

(2) for $m_2 = -1$

$\Rightarrow c_2 = -\left[\frac{-1-1}{-6+2+2}\right] \Rightarrow c_2 = -\left[\frac{-2}{-2}\right] = -1$

$Y = m_2x + c_2 \Rightarrow Y = -x-1 \Rightarrow \boxed{Y+x+1=0}$

(3) for $m_3 = -1/2$

$\Rightarrow c_3 = -\left[\frac{-\frac{1}{2}-\frac{1}{4}}{-6\times\frac{1}{4}+2+2\times\frac{1}{2}}\right] \Rightarrow c_3 = -\left[\frac{\frac{-4-2}{8}}{\frac{-3+4+2}{2}}\right]$

$\Rightarrow c_3 = -\left[\frac{-3}{4}\times\frac{2}{3}\right] \Rightarrow \boxed{c_3 = \frac{1}{2}}$

$Y = m_3x + c_3 \Rightarrow Y = -\frac{1}{2}x + \frac{1}{2}$

$\Rightarrow 2Y = -x+1 \Rightarrow \boxed{2y+x-1=0}$

Given eqⁿ of curve -

$x^3-2y^3+2x^2y-xy^2+xy-y^2+1=0$ — (1)

Multiplying all asymptote

$\Rightarrow (Y-x)(Y+x+1)(2y+x-1) = 0$ — (2)

Subtracting both eqn (2 − 1)

$$x^3 - 2y^3 + 2x^2y - xy^2 + yx - y^2 + 1 = 0$$
$$(Y-x)(Y+x+1)(2y+x-1) = 0$$
$$x - y + 1 = 0$$

which is straight line cut the curve at $n(n-2)$ i.e. $3(3-2) = 3$ points

Inspection Method.

Ques1) $\underbrace{x^3 - 2y^3 + xy(2x-y)}_{} + x + y + 7 = 0$

Sol $\underbrace{x^3 - 2y^3 + 2x^2y - xy^2}_{} + \underbrace{x + y + 7}_{} = 0$

$$F_3 + F_1 = 0$$

Putting $f_3 = 0$

$\Rightarrow x^3 - 2y^3 + 2x^2y - xy^2 = 0$

$\Rightarrow 2y^3 + xy^2 - x^3 - 2x^2y = 0$

$\Rightarrow y^2(2y + x) - x^2(x + 2y) = 0$

$\Rightarrow (y^2 - x^2)(2y + x) = 0$

$\Rightarrow y + x = 0, \; y - x = 0, \; 2y + x = 0$ ans//

Ques2) $(x^2 - 3x + 2)(x + y - 2) + 1 = 0$

Sol $\underbrace{(x-1)(x-2)(x+y-2)}_{} + 1 = 0$

$$F_3 + F_0 = 0$$

$\Rightarrow F_3 = 0$

$(x-1)(x-2)(x+y-2) = 0$

$\Rightarrow x - 1 = 0, \; x - 2 = 0, \; x + y - 2 = 0$

Ques3) $(\alpha_1 x + \beta_1 y + \gamma_1)(\alpha_2 x + \beta_2 y + \gamma_2) + \gamma_3 = 0$

Soln $F_2 + F_0 = 0$

$F_2 = 0$

$\Rightarrow (\alpha_1 x + \beta_1 y + \gamma_1)(\alpha_2 x + \beta_2 y + \gamma_2) = 0$

$\Rightarrow \alpha_1 x + \beta_1 y + \gamma_1 = 0,$

$\alpha_2 x + \beta_2 y + \gamma_2 = 0$ ans//

Ques4) $(Y-2x)^2(3x+4y) + 3(Y-2x)(3x+4y) - 5 = 0$

Soln $(Y-2x)(3x+4y)[(Y-2x) + 3] - 5 = 0$

$F_3 = 0$

$\Rightarrow (Y-2x)(3x+4y)(Y-2x) = 0$

$\Rightarrow Y - 2x = 0, \; 3x + 4y = 0, \; Y - 2x = 0$

Asymptote parallel to the coordinate axis

→ Asymptote ∥ to x axis:- This can be obtained by equating to zero the coefficient of higher power of x in equation of curve, provided it is not a constant term

Ques1) Finding the asymptote of the curve, which is ∥ to x axis

$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) = 0$$

Soln $x^2y^2 - a^2x^2 - a^2y^2 - a^3x - a^3y = 0$

$\Rightarrow x^2(y^2 - a^2) - a^2y^2 - a^3x - a^3y = 0$

Putting coefficient of highest power of $x=0$

$\Rightarrow (y^2-a^2)=0 \Rightarrow y=+a, y=-a$ are parallel asymptote about X axis

→ Asymptote ∥ to Y axis:- This can be done by equating to 0. The coefficient of highest power of y in eq^n of curve. provided it is not a constant term

Ques1) finding asymptotes of the curve which are ∥ to y axis

$x^2y^2 - a^2(x^2-y^2) - a^3(x+y)=0$

Sol^n $x^2y^2 - a^2x^2 - a^2y^2 - a^3x - a^3y = 0$

$y^2(x^2-a^2) - a^2x^2 - a^3x - a^3y = 0$

$\Rightarrow x^2-a^2=0$

$\Rightarrow x=+a, x=-a$ are ∥ asymptote about y axis

UNIT-3

Mean Value Theorems

Rolle's Theorem:-

Statement: If $f(x)$ is a function define on $[a,b]$

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

(iii) $f(a)=f(b)$

Then $\exists$ atleast one point $c\in(a,b)$ such that $f'(c)=0$

Ques1) Verify Rolle's theorem in $[-1,1]$ for the function $f(x)=x^2$

Sol^n We know that $f(x)=x^2$ is differentiable and continuous on $[-1,1]$ and $f(1)=1=f(-1)$

Then $f'(x)=(x^2)'=2x$ exist in $[-1,1]$

Then $c\in(-1,1)$ such that $f'(c)=0$

$\Rightarrow 2c=0 \Rightarrow c=0\in(-1,1)$

Hence Rolle's theorem is verified

Ques2) Let $f(x,y)=xy^2-x^2y$ find the proper value of θ of mean value theorem if $a=0, b=0, h=1, k=2$

Sol^n $f(x,y)=xy^2-x^2y$

$\Rightarrow f(a,b)=f(0,0)=0$

$\Rightarrow f(a+h, b+k) \Rightarrow f(1,2) = 1\times2^2 - 1^2\times2$

$\Rightarrow f(1,2)=4-2 \Rightarrow \boxed{f(1,2)=2}$

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$\Rightarrow \frac{\partial f}{\partial x}(x,y) = y^2 - 2xy$

$\Rightarrow \frac{\partial f}{\partial y}(x,y) = 2xy - x^2$

Thus at $(a+\theta h,\ b+\theta k)$ i.e at $(\theta, 2\theta)$

By Mean value theoram

$$f(a+h, b+k) - f(a,b) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a+\theta h, b+\theta k)$$

$$\Rightarrow f(1,2) - f(0,0) = \left[\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right] f(\theta, 2\theta)$$

$$\left\{\begin{array}{l} \because \frac{\partial f}{\partial x}(f(\theta,2\theta) = 4\theta^2 - 4\theta^2 = 0 \\ \because \frac{\partial f}{\partial y} f(\theta,2\theta) = 4\theta^2 - \theta^2 = 3\theta \end{array}\right\}$$

$\Rightarrow \cancel{2} = \cancel{2} \times 3\theta^2 \Rightarrow \theta^2 = 1/3$

$\Rightarrow \theta = \pm 1/\sqrt{3}$

Here $\theta = 1/\sqrt{3}$ is the required value because $0 < 1/\sqrt{3} < 1$

Ques3) Let $f(x,y) = xy^2 - x^2y$ find the proper value of θ of mean value theoram if

i) $a = b = 0,\ h = 3,\ k = 2$

Soln

$f(x,y) = xy^2 - x^2y$

$f(a,b) = f(0,0) = 0$

$f(a+h, b+k) = f(3,2) = 3\times4 - 9\times2$

$\Rightarrow f(3,2) = 12 - 18 \Rightarrow f(3,2) = -6$

$\Rightarrow \frac{\partial f}{\partial x}(x,y) = y^2 - 2xy$, $\frac{\partial f}{\partial y}(x,y) = 2xy - x^2$

at $(a+\theta h,\ b+\theta k)$ i.e $(3\theta, 2\theta)$

$\Rightarrow \frac{\partial f}{\partial x}(3\theta, 2\theta) = 4\theta^2 - 2\times3\theta\times2\theta$

$\Rightarrow \frac{\partial f}{\partial x}(3\theta,2\theta) = 4\theta^2 - 12\theta^2 \Rightarrow \boxed{\frac{\partial f}{\partial x}(3\theta,2\theta) = -8\theta^2}$

$\Rightarrow \frac{\partial f}{\partial y}(3\theta,2\theta) = 2\times3\theta\times2\theta - 9\theta^2$

$\Rightarrow \boxed{\frac{\partial f}{\partial y}(3\theta,2\theta) = 3\theta^2}$

By mean value theoram

$$f(a+h, b+k) - f(a,b) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a+\theta h, b+\theta k)$$

$$\Rightarrow f(3,2) - f(0,0) = \left[3\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right] f(3\theta, 2\theta)$$

$\Rightarrow -6 = 3\times(-8\theta^2) + 2\times3\theta^2$

$\Rightarrow -6 = -24\theta^2 + 6\theta^2 \Rightarrow +\cancel{6} = +\overset{3}{\cancel{18}}\theta^2$

$\Rightarrow \theta^2 = 1/3 \Rightarrow \theta = \pm 1/\sqrt{3}$

$\theta = 1/\sqrt{3}$ is required value

(ii) $a = b = 1,\ h = 3,\ k = 2$

$f(x,y) = xy^2 - x^2y$

$f(a,b) = 0$

$f(a+h, b+k) = f(4,3) = 4\times9 - 16\times3$

$\Rightarrow f(4,3) = 36 - 48 \Rightarrow f(4,3) = -12$

$\Rightarrow \frac{\partial f}{\partial x}(x,y) = y^2 - 2xy$, $\frac{\partial f}{\partial y}(x,y) = 2xy - x^2$

at $(a+\theta h, b+\theta k)$ i.e $(1+3\theta, 1+2\theta)$

$\Rightarrow \frac{\partial f}{\partial x}(1+3\theta, 1+2\theta) = (1+2\theta)^2 - 2\times(1+3\theta)(1+2\theta)$

$\Rightarrow \frac{\partial f}{\partial x}(1+3\theta, 1+2\theta) = 1+4\theta^2+4\theta - (2+6\theta)(1+2\theta)$

$= 1+4\theta^2+\cancel{4\theta} - 2-\cancel{4\theta}-6\theta-12\theta^2$

$\Rightarrow \frac{\partial f}{\partial x}(1+3\theta, 1+2\theta) = -1-6\theta-8\theta^2$

$\Rightarrow \frac{\partial f}{\partial y}(1+3\theta, 1+2\theta) = 2(1+3\theta)(1+2\theta) - (1+3\theta)^2$

$= 2+4\theta+6\theta+12\theta^2 - [1+9\theta^2+6\theta]$

$= 2+4\theta+\cancel{6\theta}+12\theta^2-1-9\theta^2-\cancel{6\theta}$

$\Rightarrow \frac{\partial f}{\partial y}(1+3\theta, 1+2\theta) = 1+4\theta+3\theta^2$

By mean value theoram

$\Rightarrow f(a+h, b+k) - f(a,b) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a+\theta h, b+\theta k)$

$\Rightarrow f(4,3) - f(1,1) = \left[3\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right] f(a+\theta h, b+\theta k)$

$\Rightarrow -12 - 0 = -3\times(1+6\theta+8\theta^2) + 2\times(1+4\theta+3\theta^2)$

$\Rightarrow 12 = 3+18\theta+24\theta^2+2+8\theta+6\theta^2$

$\Rightarrow 9 = 26\theta+30\theta^2+2$

$\Rightarrow 30\theta^2+26\theta-7$ WRONG

$\Rightarrow -12 = -3-18\theta-24\theta^2+2+8\theta+6\theta^2$

$\Rightarrow -12 = -1-10\theta-18\theta^2$

$\Rightarrow 18\theta^2+10\theta-11=0$

$\Rightarrow \frac{-10 \pm \sqrt{100+4\times18\times11}}{2\times18}$

$\Rightarrow \frac{-10\pm\sqrt{100+792}}{36} \Rightarrow \frac{-10\pm\sqrt{892}}{36}$

$\Rightarrow \frac{-10\pm 29.866}{36} \Rightarrow \frac{19.866}{36}$

$\Rightarrow$ approx $\theta = 5/9$

Partial Differentiation

Ques1) Prove that $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$ where $f(x,y) = x^3y + e^{xy^2}$

Sol $\frac{\partial f}{\partial x} = 3x^2y + e^{xy^2}y^2$, $\frac{\partial f}{\partial y} = x^3 + e^{xy^2}2xy$

$\Rightarrow \frac{\partial^2 f}{\partial x\partial y} = 3x^2 + e^{xy^2}2y + y^2e^{xy^2}2xy$

$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x\partial y} = 3x^2 + 2ye^{xy^2} + 2xy^3e^{xy^2}}$

$\Rightarrow \frac{\partial^2 f}{\partial y\partial x} = 3x^2 + e^{xy^2}2y + 2ye^{xy^2}y^2$

$\Rightarrow \boxed{\frac{\partial^2 f}{\partial y\partial x} = 3x^2+2ye^{xy^2}+2y^3e^{xy^2}}$

$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$ Hence proved

Ques2) If $u = \tan^{-1}(y/x) + \sin^{-1}(x/y)$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Sol^n $\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2} \cdot y\left[\frac{-1}{x^2}\right] + \frac{1}{\sqrt{1-(x/y)^2}} \cdot \frac{1}{y}$ — (1)

$\Rightarrow \frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} + \frac{1}{\sqrt{1-(x/y)^2}} \cdot x\left[\frac{-1}{y^2}\right]$ — (2)

$\left\{\because x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0\right\}$

Multiplying x and y in eq^n (1) and (2) respectively

$\Rightarrow x\frac{\partial u}{\partial x} = x\left[\frac{x^2}{x^2+y^2}\, y\left(\frac{-1}{x^2}\right) + \frac{1}{\sqrt{1-(x/y)^2}} \cdot \frac{1}{y}\right]$

$\Rightarrow x\frac{\partial u}{\partial x} = -\frac{y}{x} \cdot \frac{x^2}{x^2+y^2} + \frac{x}{y} \cdot \frac{1}{\sqrt{1-(x/y)^2}}$ — (3)

$\Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{x} \cdot \frac{x^2}{x^2+y^2} + \frac{-x}{y} \cdot \frac{1}{\sqrt{1-(x/y)^2}}$ — (4)

adding both the eq^n (3) and (4)

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{-y}{x} \cdot \frac{x^2}{x^2+y^2}} + \cancel{\frac{x}{y} \cdot \frac{1}{\sqrt{1-(x/y)^2}}} + \cancel{\frac{y}{x} \cdot \frac{x^2}{x^2+y^2}} - \cancel{\frac{x}{y} \cdot \frac{1}{\sqrt{1-(x/y)^2}}}$

$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}$ Hence Proved.

Ques 3) If $u = ax^2 + 2hxy + by^2$, find $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$

Sol^n $u = ax^2 + 2hxy + by^2$

$\Rightarrow \frac{\partial u}{\partial x} = 2ax + 2hy \quad \Rightarrow \boxed{\frac{\partial^2 u}{\partial x \partial y} = 2h}$

$\Rightarrow \frac{\partial u}{\partial y} = 2by + 2hx \quad \Rightarrow \boxed{\frac{\partial^2 u}{\partial y \partial x} = 2h}$

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 2h$

Ques 4) If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Sol^n $\frac{\partial u}{\partial x} = f'\left(\frac{-1}{x^2}\right) + f'\frac{y}{x}$

$\Rightarrow \frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \Rightarrow \frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\frac{1}{x}$

$\Rightarrow x\frac{\partial u}{\partial x} = x\left[f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)\right] \Rightarrow x\frac{\partial u}{\partial x} = \frac{-y}{x} f'\frac{y}{x}$

$\Rightarrow y\frac{\partial u}{\partial y} = y\left[f'\frac{y}{x} \cdot \frac{1}{x}\right] \Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{x} f'\frac{y}{x}$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{-y}{x} f'\frac{y}{x}} + \cancel{\frac{y}{x} f'\frac{y}{x}}$

$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}$ ans// Hence proved

Ques 5) If $u = (1-2xy+y^2)^{-1/2}$ then prove that $\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 0$

Sol^n $u = (1-2xy+y^2)^{-1/2}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{-1}{2}(1-2xy+y^2)^{-3/2}(-2y)$

$\left\{\because u = (1-2xy+y^2)^{-1/2}\right\}$

$\Rightarrow \frac{\partial u}{\partial x} = yu^3$ — (1)

$\Rightarrow \frac{\partial u}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2x+2y)$

$\Rightarrow \frac{\partial u}{\partial y} = +\frac{1}{2}(1-2xy+y^2)^{-3/2} + 2(x-y)$

$\Rightarrow \frac{\partial u}{\partial y} = (x-y)u^3$ — (2)

Now $\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} = \frac{\partial}{\partial x}(1-x^2)\,yu^3$

$\Rightarrow y(-2x)u^3 + y(1-x^2)3u^2\frac{\partial u}{\partial x}$

$\Rightarrow y(-2x)u^3 + y(1-x^2)3u^2\,yu^3$

$\Rightarrow -2xyu^3 + 3y^2u^5(1-x^2)$ — (3)

Also $\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = \frac{\partial}{\partial y}y^2(x-y)u^3$

$\Rightarrow \frac{\partial}{\partial y}(y^2x-y^3)u^3 \Rightarrow (2xy-3y^2)u^3 + (y^2x-y^3)3u^2\frac{\partial u}{\partial y}$

$\Rightarrow (2xy-3y^2)u^3 + y^2(x-y)\,3u^2(x-y)u^3$

$\Rightarrow (2xy-3y^2)u^3 + y^2(x-y)^2 3u^5$

$\Rightarrow 2xyu^3 + 3y^2u^5[(x-y)^2 - u^{-2}]$

$\Rightarrow 2xyu^3 + 3y^2u^5[(x-y)^2 - (1-2xy+y^2)]$

$\{\because u^{-2} = 1-2xy+y^2\}$

$\Rightarrow 2xyu^3 + 3y^2u^5[x^2+\cancel{y^2}-\cancel{2xy}-1+\cancel{2xy}-\cancel{y^2}]$

$\Rightarrow 2xyu^3 + 3y^2u^3(x^2-1)$

$\Rightarrow 2xyu^3 - 3y^2u^3(1-x^2)$ any//

Euler's theorem :-

If $u(x)$ is homogeneous function of degree n in x & y then P.I

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Hence degree $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Proof :- Let u is a homogenous function of order n i.e $u = x^n f(y/x)$

diff. w. r. to x

$\Rightarrow \frac{\partial u}{\partial x} = x^n f'(y/x)\left(-\frac{y}{x^2}\right) + nx^{n-1}f\left(\frac{y}{x}\right)$

$\Rightarrow x\frac{\partial u}{\partial x} = -\frac{y}{x}x^n f'\left(\frac{y}{x}\right) + nx^{n-1}f\left(\frac{y}{x}\right)$

diff w. r to y

$\Rightarrow \frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x} + \cancel{x^{n-1}}f\left(\frac{y}{x}\right)$

$\Rightarrow y\frac{\partial u}{\partial y} = y\,x^{n-1}f'\left(\frac{y}{x}\right)$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n x^n f(y/x)$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

again diff w. r to x

$\Rightarrow \left[x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right] + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x}$

multiply by x

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = nx\frac{\partial u}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = nx\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = (n-1)x\frac{\partial u}{\partial x}$$

Similarly diff. w r to y and multiply by y

$$\Rightarrow y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x\partial y} = (n-1)y\frac{\partial u}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left[nu\right]$$

Ques1) If $u = \sin^{-1}\left[\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right]$

PT $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u$

Soln $\sin u = \left[\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right]$

Let $\sin u = z$

$$\Rightarrow z = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$$

By Euler's theorm :-

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left(\frac{1}{4} - \frac{1}{5}\right)z$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \left(\frac{1}{20}\right)\sin u$$

$$\Rightarrow \cos u\, x\frac{\partial u}{\partial x} + \cos u\, y\frac{\partial u}{\partial y} = \frac{1}{20}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\frac{\sin u}{\cos u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u$$

Ques2) If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

Soln PT - $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

Soln $\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

let $\cos u = z$

$$\Rightarrow z = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

By Euler's theorem

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left(1 - \frac{1}{2}\right)\cos u$$

$$\Rightarrow x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

$$\Rightarrow -\sin u\, x\frac{\partial u}{\partial x} - \sin u\, y\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\Rightarrow \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}\frac{\cos u}{-\sin u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

Hence proved

Ques 3) $u = e^{\frac{x^3+y^3}{3x-4y}}$ P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$

Soln taking log both side

$\Rightarrow \log u = \frac{x^3+y^3}{3x-4y}$ let $\log u = z$

$\Rightarrow z = \frac{x^3+y^3}{3x-4y}$ By Euler's theoram

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (3-1)z$

$\Rightarrow x\frac{\partial}{\partial x}\log u + y\frac{\partial}{\partial y}\log u = 2\log u$

$\Rightarrow \frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y} = 2\log u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ Hence proved

Ques 4) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

PT (1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$

Soln $\tan u = \frac{x^3+y^3}{x+y}$ let $\tan u = z$

$z = \frac{x^3+y^3}{x+y}$ By Euler's theoram

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (3-1)z$

$\Rightarrow x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}\tan u = 2\tan u$

$\Rightarrow x\frac{\partial u}{\partial x}\sec^2 u + y\frac{\partial u}{\partial y}\sec^2 u = 2\tan u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}\times\cos^2 u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Hence Proved

Diff. w. r. to x

$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x}$

$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u - 1)\frac{\partial u}{\partial x}$

$\Rightarrow$ Multiply by x

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u-1)x\frac{\partial u}{\partial x}$ — (1)

Similarly diff. w.r.to y and multiply by y

$\Rightarrow y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u-1)y\frac{\partial u}{\partial y}$ — (2)

adding eqn (1) & 2

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2\cos 2u-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2\cos 2u-1)\sin 2u$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [(2(1-2\sin^2 u)-1)\sin 2u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$$

Hence proved

$$\left\{\begin{aligned} \because \cos 2u &= 2\cos^2 u - 1 \\ \text{AND } \cos 2u &= 1 - 2\sin^2 u \end{aligned}\right\}$$

Quess) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Solⁿ $\tan u = \frac{x^3+y3}{x-y}$ let $\tan u = z$

$$z = \frac{x^3+y^3}{x-y}$$

By euler's theoram

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (3-1)\tan u$$

$$x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}\tan u = 2\tan u$$

$$x\frac{\partial u}{\partial x}.\sec^2 x + y\frac{\partial u}{\partial y}\sec^2 x = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u} \times \cos^2 u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Hence proved

Taylor's theoram for function of Two variable

If $f(x,y)$ and all its partial derivatives upto n^{th} order are infinite and continuous for all point (x,y)

Then
$$\boxed{f(a+h, b+b) = f(a,b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f(a,b) + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a,b) + \cdots}$$

Another form:-

$$\boxed{f(x,y) = f(a,b) + \left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}f(a,b) + \frac{1}{2!}\left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}^2 f(a,b) + \cdots}$$

OR

$$\boxed{f(x,y) = f(a,b) + \{(x-a)f_x + (y-b)f_y\} + \frac{1}{2!}\{(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\} + \cdots}$$

$$\left\{\text{here } f_{xx} = \frac{\partial^2 f}{\partial x^2} \ \& \ f_{yy} = \frac{\partial^2 f}{\partial y^2}\right\}$$

Putting $a=0$, $b=0$ in the above series

$$\boxed{f(x,y) = f(0,0) + \{x f_x + y f_y\} + \frac{1}{2!}\{x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\} + \cdots}$$

↳ Maclaurins for two variable

Ques1) Expand $e^x \sin y$ in powers of x & y as far as term of third degree

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Sol^n $f(x,y) = f(0,0) + \frac{1}{1!}[xf_x + yf_y] + \frac{1}{2!}[x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}] + \frac{1}{3!}[x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}] + \dots$

$\Rightarrow f(x,y) = e^x \sin y \qquad \Rightarrow f(x) = e^x \sin y$

$\Rightarrow f_{xx} = e^x \sin y \qquad \Rightarrow f_{xxx} = e^x \sin y$

$\Rightarrow f_y = e^x \cos y \qquad \Rightarrow f_{yy} = -e^x \sin y$

$\Rightarrow f_{yyy} = -e^x \cos y$

$\Rightarrow f_{xy} = e^x \cos y \qquad \Rightarrow f_{xxy} = e^x \cos y$

$\Rightarrow f_{xyy} = -e^x \sin y$

at $[0,0]$

$\Rightarrow f(x)_0 = 0 \qquad \Rightarrow f_{xx0} = 0$

$\Rightarrow (f_{xxx})_0 = 0 \qquad \Rightarrow (f_y)_0 = 1 \qquad \Rightarrow (f_{yy})_0 = 0$

$\Rightarrow (f_{yyy})_0 = -1 \qquad \Rightarrow (f_{xy})_0 = 1 \qquad \Rightarrow (f_{xxy})_0 = 1$

$\Rightarrow (f_{xyy})_0 = 0 \qquad \Rightarrow f(0,0) = 0$

$\Rightarrow e^x \sin y = 0 + \frac{1}{1!}[x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)] + \frac{1}{6}[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)] + \dots$

$\Rightarrow e^x \sin y = y + \frac{1}{\cancel{2}!}[\cancel{2}xy] + \frac{1}{6}[3x^2y - y^3] + \dots$

$\Rightarrow e^x \sin y = y + xy + \frac{\cancel{3}x^2y}{2} - \frac{y^3}{6} + \dots$

ans//

Ques 2) Expand $e^x \log(1+y)$ in Taylor's Series around origin in term of second degree

Sol^n $f(x,y) = f(0,0) + [xf_x + yf_y] + \frac{1}{2!}[x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}] + \frac{1}{6}[x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}] + \dots$

$\Rightarrow f(x,y) = e^x \log(1+y) \qquad \Rightarrow f(0,0) = 0$

$\Rightarrow f(x) = e^x \log(1+y) \qquad \Rightarrow f(x)_0 = 0$

$\Rightarrow f(xx) = e^x \log(1+y) \qquad \Rightarrow f(xx)_0 = 0$

$\Rightarrow f(y) = \frac{e^x}{1+y} \qquad \Rightarrow f(y)_0 = 1$

$\Rightarrow f(yy) = \frac{-e^x}{(1+y)^2} \qquad \Rightarrow f(yy)_0 = -1$

$\Rightarrow f_{xy} = \frac{e^x}{1+y} \qquad \Rightarrow f(x,y)_0 = 1$

$\Rightarrow e^x \log(1+y) = 0 + [x(0) + y(1)] + \frac{1}{2}[x^2(0) + 2xy(1) + y^2(-1)] + \dots$

$\Rightarrow e^x \log(1+y) = y + \frac{1}{2}[2xy - y^2] + \dots$

$\Rightarrow e^x \log(1+y) = y + xy - \frac{y^2}{2} + \dots$

ans/

formula -

$f(x,y) = f(a,b) + [(x-a)^2 \partial/\partial x + (y-b)^2 \partial/\partial y] f(a,b) + \frac{1}{2!}[(x-a)\partial/\partial x + (y-b)\partial/\partial y]^2 f(a,b) + \dots R_n$

$R_n = \frac{1}{n!}[(x-a)\partial/\partial x + (y-b)\partial/\partial y]^n f[a+(x-a)\theta + b(y-b)\theta]$

$0 < \theta < 1$

Ques1) Expand $f(x,y) = x^2 + xy + y^2$ in powers of $(x-2)$ & $(y-3)$

Soln $a = 2,\ b = 3$

$$\boxed{f(x,y) = f(a,b) + [(x-a)f_x + (y-b)f_y] + \frac{1}{2!}[(x-a)^2 f_{xx} + (y-b)^2 f_{yy} + 2(x-a)(y-b)f_{xy}]}$$

$\Rightarrow f(x) = 2x + y \quad \Rightarrow (f_x)(2,3) = 2\times2+3 = 7$

$\Rightarrow f_y = x + 2y \quad \Rightarrow (f_y)(2,3) = 2 + 2\times3 = 8$

$\Rightarrow f_{xx} = 2 \quad \Rightarrow (f_{xx})(2,3) = 2$

$\Rightarrow f_{yy} = 2 \quad \Rightarrow (f_{yy})(2,3) = 2$

$\Rightarrow f_{xy} = 1 \quad \Rightarrow (f_{xy})(2,3) = 1$

$\Rightarrow f(2,3) = 4 + 6 + 9 = 19$

$$\Rightarrow x^2 + xy + y^2 = 19 + [(x-2)7 + (y-3)8] + \frac{1}{2}[(x-2)^2 2 + (y-3)^2 2 + 2(x-2)(y-3)]$$

$$\Rightarrow x^2 + xy + y^2 = 19 + 7(x-2) + 8(y-3) + (x-2)^2 + (y-3)^2 + (x-2)(y-3) + \dots$$ ans//

Ques2) Expand $\sin xy$ in powers of $(x-1)$ & $(y-\pi/2)$ upto second degree term

Soln $a = 1,\ b = \pi/2$

$$\Rightarrow f(x,y) = f(a,b) + [(x-a)f_x + (y-b)f_y] + \frac{1}{2!}[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}]$$

$\Rightarrow f(x) = y\cos xy \quad \Rightarrow (f_x)(1,\pi/2) = \pi/2 \cos\pi/2 = 0$

$\Rightarrow f(y) = x\cos xy \quad \Rightarrow (f_y)(1,\pi/2) = 1\cos\pi/2 = 0$

$\Rightarrow f_{xx} = -y^2 \sin xy \quad \Rightarrow (f_{xx})(1,\pi/2) = -\pi^2/4 \times 1 = -\pi^2/4$

$\Rightarrow f_{yy} = -x^2 \sin xy \quad \Rightarrow (f_{yy})(1,\pi/2) = -1\times1 = -1$

$\Rightarrow f_{xy} = \cos xy - xy\sin xy$

$\Rightarrow (f_{xy})(1,\pi/2) = \cos\frac{\pi}{2} - \frac{\pi}{2}\sin\pi/2 = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$

$f(x,y) = \sin xy \quad \Rightarrow f(1,\pi/2) = 1$

$$\Rightarrow \sin xy = 1 + [(x-1)(0) + (y-\pi/2)(0)] + \frac{1}{2}[(x-1)^2(-\pi^2/4) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)] + \dots$$

$$\Rightarrow \sin xy = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y-\pi/2) - \frac{1}{2}(y-\pi/2)^2$$ ans

Ques3) Expand $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$

Soln $a = 1,\ b = -2$

$$f(x,y) = f(a,b) + [(x-a)f_x + (y-b)f_y] + 1/2!\,[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}] + \dots$$

$\Rightarrow f(x,y) = x^2y + 3y - 2 \quad \Rightarrow f(1,-2) = -2-6-2 = -10$

$\Rightarrow f(x) = 2xy \quad \Rightarrow (f_x)(1,-2) = 2\times1\times(-2) = -4$

$\Rightarrow f_{xx} = 2y \quad \Rightarrow (f_{xx})(1,-2) = 2\times(-2) = -4$

$\Rightarrow (f_y) = 3 + x^2 \quad \Rightarrow (f_y)(1,-2) = 3\times1 = 4$

$\Rightarrow (f_{yy}) = 0 \quad \Rightarrow (f_y)(1,-2) = 0$

$\Rightarrow (f_{xy}) = 2x \quad \Rightarrow (f_{xy})(1,-2) = 2$

$\Rightarrow f_{xxx} = 0 \quad \Rightarrow (f_{xxx})(1,-2) = 0$

$\Rightarrow f_{yyy} = 0 \quad \Rightarrow (f_{yyy})(1,-2) = 0$

$\Rightarrow f_{xxy} = 2 \quad \Rightarrow (f_{xxy})(1,-2) = 2$

$\Rightarrow f_{xyy} = 0 \quad \Rightarrow (f_{xyy})(1,-2) = 0$

$$\Rightarrow x^2y+3y-2 = -10+[(x-1)(-4)+(y+2)(4)] + \frac{1}{2}[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2 0]$$

$$+\frac{1}{6}[(x-1)^3 0 + 3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2 0 + (y+2)^3 0]$$

$$\Rightarrow x^2y+3y-2 = -10-4(x-1)+4(y+2)-2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

ans//

Maxima & Minima of function of two Independent variable

$\Rightarrow$ Condition for Maxima & Minima

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0, \quad \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0$$

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}, \quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}, \quad s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}$$

1) If $rt-s^2>0$, $r>0$ $f(x,y)$ is minimum at (a,b)

2) If $rt-s^2>0$, $r<0$, $f(x,y)$ is maximum at (a,b)

3) If $rt-s^2<0$, Neither Minma Nor maxima

4) If $rt-s^2=0$, doubtful case

Ques1) Discuss the Maxima & Minima of $x^2+y^2+6x+12$

Soln $f(x,y) = x^2+y^2+6x+12$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x+6=0, \quad \frac{\partial f}{\partial y} = 2y=0$$

$\Rightarrow$ Root $2x+6=0 \Rightarrow \boxed{x=-3}$

$\Rightarrow$ $2y=0 \Rightarrow \boxed{y=0}$

$(-3,0)$ is the saddle Point

$$\Rightarrow r = \frac{\partial^2 f}{\partial x^2} = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\Rightarrow t = \frac{\partial^2 f}{\partial y^2} = 2$$

Points	r	s	t	$rt-s^2$	Result
$(-3,0)$	2	0	2	4	Minima

for minimum value put $x=-3, y=0$

$\Rightarrow (-3)^2+(0)^2+6\times(-3)+12$

$\Rightarrow 9-18+12 \Rightarrow 3$ is the Minimum value

Ques2) find maxima & minima of curve $u = x_1^3+x_2^3+x_1^2+4x_2^2+6$

Soln $\frac{\partial u}{\partial x_1} = 3x_1^2+2x_1 = 0$

$\frac{\partial u}{\partial x_2} = 3x_2^2 + 8x_2 = 0$

$\Rightarrow 3x_1^2+2x_1=0 \Rightarrow x_1(3x_1+2)=0$

$\Rightarrow \boxed{x_1=0}, \boxed{x_1=-2/3}$

$\Rightarrow 3x_2^2+8x_2=0 \Rightarrow x_2(3x_2+8)=0$

$\Rightarrow \boxed{x_2=0}, \boxed{x_2=-8/3}$

Req Points :- $(0,0), (-2/3, -8/3)$

$r = \frac{\partial^2 u}{\partial x_1^2} = 6x_1 + 2$, $s = \frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$

$t = \frac{\partial^2 u}{\partial x_2^2} = 6x_2 + 8$

Point	r	t	s	$rt - s^2$	Result
(0,0)	2	8	0	16	Minima
(-2/3, -8/3)	-2	-8	0	16	Maxima

Minimum value :- $x_1^3 + x_2^3 + x_1^2 + 4x_2^2 + 6$

at $(0,0) = 0+0+0+0+6 = 6$

Maximum value :- $-\frac{8}{27} - \frac{512}{27} + \frac{4}{9} + \frac{4\times 64}{9} + 6$

$\Rightarrow \frac{-520}{27} + \frac{256}{9} + \frac{4}{9} + 6 \Rightarrow \frac{-520}{27} + \frac{260}{9} + 6$

$\Rightarrow \frac{-520 + 780 + 162}{27} \Rightarrow \frac{422}{27}$

Ques 3) find max & min of $u = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Solⁿ $\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 30x + 72 = 0$ — (1)

$\frac{\partial u}{\partial y} = 6xy - 30y = 0$ — (2)

$r = \frac{\partial^2 u}{\partial x^2} = 6x - 30$

$s = \frac{\partial^2 u}{\partial x \partial y} = 6y$, $t = \frac{\partial^2 u}{\partial y^2} = 6x - 30$

from eqⁿ (2)

$\Rightarrow 6y(x - 5) = 0 \Rightarrow \boxed{y=0}, \boxed{x=5}$

putting $y=0$ in eqⁿ (1)

$\Rightarrow 3x^2 - 30x + 72 = 0$

$\Rightarrow 3[x^2 - 10x + 24] = 0 \Rightarrow x^2 - 10x + 24 = 0$

$\Rightarrow x^2 - 6x - 4x + 24 = 0$

$\Rightarrow x(x-6) - 4(x-6) = 0$

$\Rightarrow (x-4)(x-6) = 0 \Rightarrow \boxed{x=4}, \boxed{x=6}$

Point (4,0) & (6,0)

$\Rightarrow$ Putting $x=5$ in eqⁿ (1)

$\Rightarrow 3\times 25 + 3y^2 - 150 + 72 = 0$

$\Rightarrow 75 + 3y^2 - 150 + 72 = 0$

$\Rightarrow 3y^2 - 150 + 147 = 0 \Rightarrow 3y^2 - 3 = 0$

$\Rightarrow 3(y^2 - 1) = 0 \Rightarrow y = \pm 1$

$\Rightarrow$ (5,1) & (5,-1) these are required

Total req points are (4,0); (6,0), (5,1) & (5,-1)

Points	r	t	s	$rt - s^2$	result
(4,0)	-6	-6	0	36	maxima
(6,0)	6	6	0	0	minima
(5,1)	0	0	6	-36	Neither Minim Nor Maxima
(5,-1)	0	0	6	-36	Neither Minim Nor Maxima

Maximum value :- at (4,0)

$\Rightarrow 64 + 0 - 240 + 288$

$\Rightarrow 64 + 48 \Rightarrow 112$ is maximum value

Minimum value at (6,0)

$216 + 0 - 520 + 432 \Rightarrow 648 - 520$

128 Minimum Value

Ques 4) find Maxima & Minima of

$$u = \sin x \sin y \sin(x+y)$$

Soln {short trick = the point in this type of cases are $(\pi/3, \pi/3)$ }

$\Rightarrow \frac{\partial u}{\partial x} = \sin y\,[\sin x \cos(x+y) + \cos x \sin(x+y)]$

$\because \sin A \cos B + \cos A \sin B = \sin(A+B)$

$\Rightarrow \frac{\partial u}{\partial x} = \sin y\,[\sin(x + x + y)]$

$\Rightarrow \frac{\partial u}{\partial x} = \sin y\,[\sin(2x+y)]$ FOF — (1)

$\Rightarrow \frac{\partial u}{\partial y} = \sin x\,[\sin y \cos(x+y) + \cos y \sin(x+y)]$

$\Rightarrow \frac{\partial u}{\partial y} = \sin x\,[\sin(2y+x)] = 0$ — (2)

from eq^n (1)

$\sin y = 0$ & $\sin(2x+y) = 0$

$\Rightarrow y = 0$ | $2x+y = \pi$ $\{\because \sin\pi = 0\}$

from eq^n (2)

$\sin x = 0$ | $\sin(2y+x) = 0$

$x = 0$ | $2y + x = \pi$

required points :- $(0,0), (\pi/3, \pi/3)$

$r = \frac{\partial^2 u}{\partial x^2} = 2\sin y \cos(x+y)$

$s = \frac{\partial^2 u}{\partial x \partial y} = \cos x \sin(2y+x) + \sin x \cos(2y+x)$

$\Rightarrow s = \frac{\partial^2 u}{\partial x \partial y} = \sin(2x+2y)$ $\{\because \sin(A+B) = \sin A \cos B + \cos A \sin B$

$\Rightarrow t = \frac{\partial^2 u}{\partial y^2} = 2\sin x\,\{\cos(2y+x)\}$

Points	r	t	s	$rt - s^2$	result
$(0,0)$	0	0	0	0	doubtful
$(\pi/3, \pi/3)$	$-\sqrt{3}$	$-\sqrt{3}$	$\frac{-\sqrt{3}}{2}$	$\frac{9}{4} > 0$	Maxima

Maximum value :- $\sin x \sin y \sin(x+y)$

$\Rightarrow \sin\frac{\pi}{3} \cdot \sin\frac{\pi}{3} \sin(2\pi/3)$

$\Rightarrow \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \Rightarrow \frac{3\sqrt{3}}{8}$ am maximum Value

NOTE :- $\cos(2\pi/3) = \cos(\pi) = (-1)$

Ques 5) Discuss the maximum and minimum value of the following function

$$u = x^3 y^2 (1 - x - y)$$

Soln $u = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial u}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ — (1)

$\frac{\partial u}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$ — (2)

Multiplying by x and y in eq^n (1) & (2) respectively

from eq^n (1) :- $3x^3y^2 - 4x^4y^2 - 3x^3y^3$ — (3)

from eq^n (2) :- $2x^3y^2 - 2x^4y^2 - 3x^3y^3$ — (4)

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(3) − (4)

$$\Rightarrow 3x^3y^2 - 4x^4y^2 - 3x^3y^3 = 0$$
$$2x^3y^2 - 2x^4y^2 - 3x^3y^3 = 0$$
$$- \quad + \quad +$$
$$x^3y^2 - 2x^4y^2 = 0$$
$$\Rightarrow x^3y^2[1-2x] = 0$$
$$\Rightarrow x^3y^2 = 0 \quad | \quad 2x = 1$$
$$\Rightarrow x = 0 \quad | \quad x = 1/2$$

Putting $x = 1/2$ in eq^n (1)

$$\Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$
$$\Rightarrow 3\times\frac{1}{4}y^2 - 4\times\frac{1}{8}y^2 - 3\times\frac{1}{4}y^3$$
$$\Rightarrow \frac{6y^2 - 4y^2 - 6y^3}{8} \Rightarrow \frac{2y^2 - 6y^3}{8}$$
$$\Rightarrow \frac{2y^2[1-3y]}{8} = 0 \Rightarrow \boxed{y = 1/3}$$

$$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$r \text{ at } (1/2, 1/3) :- 6\times\frac{1}{2}\times\frac{1}{9} - 12\times\frac{1}{4}\times\frac{1}{9} - 6\times\frac{1}{2}\times\frac{1}{27}$$

$$\Rightarrow \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{9} \Rightarrow \boxed{-1/9}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

$$\text{at } (1/2, 1/3) = 6\times\frac{1}{4}\times\frac{1}{3} - 8\times\frac{1}{8}\times\frac{1}{3} - 9\times\frac{1}{4}\times\frac{1}{9}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$$

$$\Rightarrow \frac{6-4-3}{12} \Rightarrow \boxed{-\frac{1}{12}}$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^4 - 2x^3 - 6x^3y$$

$$\text{at } (1/2, 1/4) = 2\times\frac{1}{16} - 2\times\frac{1}{8} - 6\times\frac{1}{8}\times\frac{1}{3}$$

$$\Rightarrow \frac{1}{8} - \frac{1}{4} - \frac{1}{4} \Rightarrow \boxed{-\frac{1}{8}}$$

$$\Rightarrow rt - s^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{72} > 0 \text{ +ve}$$

function is minimum

minimum value at $(1/2, 1/3)$ of u :-

$$u = x^3y^2 - x^4y^2 - x^3y^3$$
$$\Rightarrow u = \frac{1}{8}\times\frac{1}{9} - \frac{1}{16}\times\frac{1}{9} - \frac{1}{8}\times\frac{1}{27}$$
$$\Rightarrow u = \frac{1}{8}\times\frac{1}{9}\left[1 - \frac{1}{2} - \frac{1}{3}\right] = \frac{1}{72}\left[\frac{6-3-2}{6}\right]$$
$$\Rightarrow u = \frac{1}{72}\times\frac{1}{6} \Rightarrow u = \frac{1}{432}$$

ans//

Sequence & Series

If terms are separated by commas then we call it Sequence

$\sum_{n=1}^{\infty} u$ or $\sum u_n$ it is the symbol used to show the infinite series

⇒ When the sum of the series is finite then we call it convergence

⇒ When the sum of the series is infinite then we call it Divergence

⇒ every finite series is convergent

Bounded Sequence :- These are of two types

- Bounded below = In these type of sequence we know the lower limit but we didn't know the higher limit ex:- $<1, \sqrt{2}, 3, \ldots \infty>$
- Bounded above = In these type of sequences we know the upper limit but didn't know the lower limit ex:- $<s_n> = <-1^2, -2^2, -3^2>$
- When the sequence is both above & below then these are bounded sequence
  ex:- $<s_n> = <1/n>$

⇒ Convergent Sequence:-

$\lim_{n\to\infty} U_n =$ finite/unique

ex:- $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots$

general term $u_n = \frac{n}{n+1}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} \Rightarrow \lim_{n\to\infty} \frac{\cancel{n}(1)}{\cancel{n}(1+1/n)}$

$\Rightarrow \lim_{n\to\infty} u_n = 1$ Hence it shows that it is convergent

⇒ Divergent Sequence.

$\lim_{n\to\infty} u_n = +\infty$ or $-\infty$

ex:- $1, 2, 3, \ldots$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n = \infty$ Hence it is convergent

⇒ Oscillating sequence:-

$\lim_{n\to\infty} u_n =$ Not unique

the result neither tend to plus or finite, infinity nor definite or neither convergent nor divergent

- $\sum_{n=1}^{\infty} u_n$ is convergent when the sequence of partial sum $\{s_n\}_{n=1}^{\infty}$ is convergent i.e $\lim_{n\to\infty} s_n =$ infinite
- $\sum_{n=1}^{\infty} u_n$ is divergent when the sequence of partial sum $\{s_n\}_{n=1}$ is divergent i.e $s_n = \infty$ or $-\infty$
- $\sum_{n=1}^{\infty} u_n$ is oscillating when the sequence of partial sum $\{s_n\}_{n=1}$ is oscillating i.e $\{s_n\}_{n=1}$ is non convergent and non properly divergent i.e oscillating b/w 2 numbers

ex of oscillating sequence :- $-1, 1, -1, 1, -1$

Theoram :- The Geometric sum $1+r+r^2+\infty$

1) convergent if $|r|<1$
2) Divergent if $r \geq 1$
3) oscillating if $r \leq -1$

ex = ① $1 + \frac{1}{2} + \frac{1}{4} + \ldots$

Here $r = 1/2 < 1$ so it is convergent

② $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3}$

$\left\{ r = \frac{2nd\ term}{1st\ term} \right\} \Rightarrow r = -1/3 < 1$ convergent

Theoram :- the infinite series Σu_n is convergent then $\lim_{n\to\infty} u_n = 0$ but its

converse is not true
This is the test for divergence

① convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$

② $\lim_{n\to\infty} u_n = 0 \not\Rightarrow$ convergent

③ $\lim_{n\to\infty} u_n \neq 0 \Rightarrow$ divergent

{ agar limit 0 nhi aati hai to vo pakka } se divergent hai

Comparision test :- If Σu_n & Σv_n be two series be +ve terms

(a) Σv_n is convergent then Σu_n is convergent if $\lim_{n\to\infty} \frac{u_n}{v_n} = k$

where k is a finite +ve non 0 number

(b) Σv_n is divergent then Σu_n is also divergent if $\lim_{n\to\infty} \frac{u_n}{v_n} = k$ where k is finite +ve number

$\Rightarrow$ P series test :- The finite series

$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$

① convergent if $P > 1$

② Divergent if $P \leq 1$

Ways to find v_n :-

① let $u_n = \frac{an^q + bn^{q-1} + \ldots}{an^p + bn^{p-1} + \ldots}$

then $v_n = \frac{1}{n^{p-q}}$ or by another way

$v_n = \frac{n^q}{n^p} \Rightarrow v_n = \frac{1}{n^{p-q}}$

② if $u_n = \frac{a}{n^k} + \frac{b}{n^{k+1}} + \ldots$ then $v_n = \frac{1}{n^k}$

ex:- Test the convergence of
$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \frac{1}{2n-1} + \ldots$

Sol: $\Sigma u_n = \sum \frac{1}{2n-1}$

let $\Sigma v_n = \Sigma 1/n$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1/2n-1}{1/n} = \frac{n}{2n-1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(2-1/n)} \Rightarrow \frac{1}{2}$$ this is finite and +ve

But $\Sigma v_n = \Sigma \frac{1}{n}$ Here $P=1$ By P series test

so the v_n is divergent due to which u_n is divergent by comparision test

Ques1) Test the convergence of the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$$

Sol^n $\Sigma u_n = \Sigma \frac{2n-1}{n(n+1)(n+2)}$

let $\Sigma v_n = \frac{n}{n^3} \Rightarrow \Sigma v_n = \Sigma \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2n-1)n^2}{n(n+1)(n+2)}$$

$$\Rightarrow \lim_{n\to\infty} = \frac{(2-1/n)}{(1+1/n)(1+2/n)} = 2$$ finite & +ve

$\Sigma v_n = \Sigma \frac{1}{n^2}$ Here $P=2$ which is greater than 1 so it is convergent

due to which Σu_n is also convergent

Ques2) Test the convergence of the series

$$\sum_{n=1}^{\infty} (\sqrt{n^4+1} - \sqrt{n^4+1})$$

Sol^n $u_n = (\sqrt{n^4+1} - \sqrt{n^4+1})$

rationalizing the denominator

$$\Rightarrow u_n = \frac{(\sqrt{n^4+1} - \sqrt{n^4-1}) \times \sqrt{n^4+1} + \sqrt{n^4-1}}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$\Rightarrow u_n = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

let $\Sigma v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2\ n^2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2\cancel{n^2}}{\cancel{n^2}\sqrt{1+1/n^4} + \sqrt{1-1/n^4}} = 2$$ finite & +ve

By P series test $\Sigma v_n = 1/n^2$ so $P = 2 > 1$

so v_n is convergent due to which u_n is convergent

Ques3) Test the convergence of series:

$$\frac{1}{n} \sin \frac{1}{n}$$

Sol^n $u_n = \frac{1}{n} \sin \frac{1}{n}$

$$\left\{ \because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \right\}$$

$$\Rightarrow u_n = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!} \cdot \frac{1}{n^3} + \ldots\right]$$

let $\Sigma v_n = \frac{1}{n^2}$ $\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} =$

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!n^3} + \ldots\right]}{1/n^2}$$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left[1 - 3 \cdot n^2\right]} + \ldots$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = 1$ finite and +ve

$\Sigma v_n = \Sigma \frac{1}{n^2}$ By P series test $p = 2 > 1$ so v_n is convergent

due to which u_n is convergent

Cauchy's Integral Root test

If for $x \geq 1$, $f(x)$ be a positive monotonic decreasing integrable function such that $f(n) = u_n$ for positive integral value of n, then $\sum_{n=1}^{\infty} u_n$ is convergent if improper integral $\int_1^{\infty} f(x)\,dx$ is convergent otherwise Σu_n is divergent

Ques 1) Use Cauchy's integral, test the convergence of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$

Sol: $\Sigma u_n = 1/n^2 = f(n)$

$\Rightarrow f(x) = 1/x^2$

$\Rightarrow \int_1^{\infty} f(x)\,dx \Rightarrow \int_1^{\infty} \frac{1}{x^2}\,dx \Rightarrow \left[\frac{x^{-2+1}}{-2+1}\right]_1^{\infty}$

$\Rightarrow -\left[\frac{1}{x}\right]_1^{\infty} \Rightarrow -\left[\cancel{\frac{1}{\infty}}^{0} - \frac{1}{1}\right]$

$\Rightarrow 1$ [finite]

$\Rightarrow \int_1^{\infty} f(x)\,dx$ is convergent so By Cauchy's integral test Σu_n is also convergent

Ques 2) Use Cauchy's integral test to show the series $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \ldots$ is convergent or divergent

Sol^n $u_n = \frac{1}{(n+1)(n+2)} = f_n$

$\Rightarrow f(x) = \frac{1}{(x+1)(x+2)}$

$\Rightarrow \int_1^{\infty} f(x)\,dx \Rightarrow \int_1^{\infty} \frac{1}{(x+1)(x+2)}\,dx$

By Partial fraction

$\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

Putting $x = -1$

$\Rightarrow A = \frac{1}{(-1+2)} \Rightarrow \boxed{A = \frac{1}{1}}$

Putting $x = -2$

$\Rightarrow B = \frac{1}{(-2+1)} \Rightarrow \boxed{B = -1}$

$\Rightarrow \int_1^{\infty} \frac{1}{(x+1)} - \frac{1}{(x+2)}\,dx \Rightarrow \left[\log(x+1) - \log x+2\right]_1^{\infty}$

$\Rightarrow \left[\log\left(\frac{x+1}{x+2}\right)\right]_1^{\infty} \Rightarrow \left[\log \frac{x}{x}\left(\frac{1+1/x}{1+2/x}\right)\right]_1^{\infty}$

$\Rightarrow \left(\log 1 - \log\frac{2}{3}\right) \Rightarrow 0 - \log\frac{2}{3}$

$\Rightarrow \int_1^{\infty} f(x)\,dx = \log\frac{3}{2}$ it is finite

so $\int_1^{\infty} f x\,dx$ is convergent by Cauchy's integral test so Σu_n is also convergent

Ques3) Show that the series $\sum_{n=1}^{\infty} 1/n^p$ is convergent if $p>1$ & divergent if $0<p\le 1$

Sol^n $\Rightarrow u_n = 1/n^p = f(n) \Rightarrow f(x) = 1/x^p$

$\Rightarrow \int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^p}dx$

$$\Rightarrow \int_1^\infty f(x)dx = \begin{cases} (\log x)_1^\infty, & p=1 \\ \left(\frac{x^{-p+1}}{-p+1}\right)_1^\infty, & p\neq 1 \end{cases}$$

$$\Rightarrow \int_1^\infty f(x)dx = \begin{cases} \infty & p=1 \\ \left(\frac{x^{1-p}}{1-p}\right)_1^\infty, & p<1 \\ \left(\frac{x^{1-p}}{1-p}\right)_1^\infty & p>1 \end{cases}$$

$$\Rightarrow \int_1^\infty f(x)dx = \begin{cases} \infty, & p<1 \\ \infty, & p=1 \\ \left(\frac{x^{1-p}}{1-p}\right)_1^\infty, & p>1 \end{cases} = \begin{cases} \infty, & p\le 1 \\ 0-\frac{1}{p-1}, & p>1 \end{cases}$$

$$\Rightarrow \int_1^\infty f(x)dx = \begin{cases} \infty & p\le 1 \\ 1/p-1, & p>1 \end{cases}$$

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p>1$ & divergent if $p\le 1$.

D'Alembert's Ratio test :-

The series Σu_n of +ve term is

① convergence if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$

② Divergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$

③ Test fails if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

Concepts - $\lim_{n\to\infty} u_n = 0$ (convergent)

$\lim_{n\to\infty} u_n > 0$ (divergent)

Ques1) Test the convergence of $1^2+2^2x+3^2x^2+\dots$ where x is +ve

Sol^n $u_n = n^2x^{n-1} \Rightarrow u_{n+1} = (n+1)^2 x^{n+1-1}$

$\Rightarrow u_{n+1} = (n+1)^2x^n$

$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{(n+1)^2x^n}{n^2x^{n-1}}$

$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 x = x$

By D'Alembert's test Σu_n is convergent if $x<1$ & divergent if $x>1$

$\Rightarrow$ if $x=1$, then $u_n = n^2$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n^2 = \infty$

at $x=1$ u_n is divergent

Hence Σu_n is convergent if $x<1$

Σu_n is divergent if $x\ge 1$

Ques 2) Test the Convergence of $\frac{x}{1\cdot2} + \frac{x^2}{3\cdot4} + \ldots$

Sol^n $\Sigma u_n = \frac{x^n}{(2n-1)2n}$ $\Rightarrow \Sigma u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$

$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+1)(2n+2)} \frac{(2n-1)(2n)}{x^n}$

$\Rightarrow \lim_{n\to\infty} \frac{(2n-1)(2n)}{(2n+1)(2n+2)} x = \lim_{n\to\infty} \frac{\cancel{n^2}(2-1/n)(2)x}{\cancel{n^2}(2+1/n)(2+2/n)}$

$\Rightarrow \lim_{n\to\infty} = x$ $\Rightarrow \boxed{\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x}$

By Alembert's ratio test
Σu_n is convergent if $x<1$ &
Σu_n is divergent if $x>1$

$\Rightarrow$ if $x=1$, then $u_n = \frac{1}{(2n-1)(2n)}$

$\Rightarrow$ let $v_n = 1/n^2$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2n-1)(2n)}{n^2}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\cancel{n^2}(2-1/n)2}{\cancel{n^2}}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = 4 > 0$

so v_n is convergent
i.e By comparision test u_n is convergent
Hence Σu_n is convergent when $x \le 1$ &
Σu_n is divergent when $x > 1$

Ques 3) Test the Convergence of $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{n^2+1}$

Sol^n $u_n = \frac{x^n}{n^2+1}$ $\Rightarrow u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^2+1} \frac{n^2+1}{x^n}$

$\Rightarrow \lim_{n\to\infty} \frac{x \, (n^2+1)}{n^2+1+2n+1}$

$\Rightarrow \lim_{n\to\infty} \frac{\cancel{n^2}[1+1/n^2]}{\cancel{n^2}[1+2/n^2+2/n]} x$

$\Rightarrow \boxed{\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x}$

By Alembert's test :-
Σu_n is convergent if $x<1$ &
Σu_n is divergent if $x>1$

$\Rightarrow$ if $x=1$, then $u_n = \frac{1}{n^2+1}$

$\Rightarrow$ let $v_n = 1/n^2$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^2+1} n^2 \Rightarrow \lim_{n\to\infty} = \frac{\cancel{n^2}}{\cancel{n^2}[1+1/n^2]}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = 1 > 0$

so v_n is convergent and by comparision test u_n is also convergent
Hence Σu_n is convergent when $x \le 1$ &
Σu_n is divergent when $x > 1$

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Cauchy's Root test of Convergence.

A series Σu_n of +ve term is

(1) Convergent if $\lim_{n\to\infty} (u_n)^{1/n} < 1$

(2) Divergence if $\lim_{n\to\infty} (u_n)^{1/n} > 1$

(3) Total fail if $\lim_{n\to\infty} (u_n)^{1/n} = 1$

Ques1) Test the Convergence of series

$$\sum_{n=1}^{\infty} \left[\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}\right]$$

Sol^n

$$\Rightarrow u_n = \left[\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}\right]$$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}\right]^{1/n}$$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{\left[\left(\frac{n+1}{n}\right)^{n^2}\right]^{1/n}}{[3^n]^{1/n}}$$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{[1+1/n]^n}{3}$$

$$\left\{\because \lim_{n\to\infty} [1+1/n]^n = e\right\}$$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \frac{e}{3} < 1$$

By cauchy's Root test Σu_n is convergent

Ques2) Test the Convergence of the series whose n^{th} term is $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

Sol^n $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$ $\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{(n^{n^2})^{1/n}}{((n+1)^{n^2})^{1/n}}$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \frac{n^n}{(n+1)^n} \Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \frac{n^n}{n^n(1+1/n)^n}$$

$$\Rightarrow \lim_{n\to\infty} u_n^{1/n} = \frac{1}{e} < 1$$

By cauchy's Root test Σu_n is convergent

Ques3) Test the Convergence of the series

$$\frac{1^3}{3} + \frac{2^3}{3^2} + \dots \frac{n^3}{3^n}$$

Sol^n $u_n = \frac{n^3}{3^n}$ $\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{(n^3)^{1/n}}{(3^n)^{1/n}}$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/3} = \lim_{n\to\infty} \frac{n^{3/n}}{3} = \lim_{n\to\infty} \frac{(n^{1/n})^3}{3}$$

$$\left\{\because \lim_{n\to\infty} n^{1/n} = 1\right\}$$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{3} < 1$$

so By cauchy's Root test the Σu_n is Convergent.

UNIT-3

Total differential constant

⇒ Total differentiation :- If $u = f(x,y)$ be function of x, y then total diffⁿ is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

⇒ diff. of composite function

let $u = f(x,y)$ where $x = f(t)$ & $y = f(t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Ques1) If $u = f(y-z, z-x, x-y)$ then PT $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let $y-z = t_1$, $z-x = t_2$, $x-y = t_3$

$u = f(t_1, t_2, t_3)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial x}$$

$\Rightarrow t_1 = y-z \Rightarrow \frac{\partial t_1}{\partial x} = 0$

$\Rightarrow t_2 = z - x = \frac{\partial t_2}{\partial x} = -1$

$\Rightarrow t_3 = x - y = \frac{\partial t_3}{\partial x} = 1$

$\Rightarrow \frac{\partial u}{\partial x} = 0 - \frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial t_3}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_3} - \frac{\partial u}{\partial t_2}$

Similarly for y & z

$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3}$ $\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_2} - \frac{\partial u}{\partial t_1}$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial t_3}} - \cancel{\frac{\partial u}{\partial t_2}} + \cancel{\frac{\partial u}{\partial t_1}} - \cancel{\frac{\partial u}{\partial t_3}} + \cancel{\frac{\partial u}{\partial t_2}} - \cancel{\frac{\partial u}{\partial t_1}}$$

$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Hence proved

Ques2) If $u = f(x/y, y/z, z/x)$

Solⁿ let $t_1 = x/y$, $t_2 = y/z$, $t_3 = z/x$

$\Rightarrow u = f(t_1, t_2, t_3)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial x}$$

$\Rightarrow t_1 = x/y \Rightarrow \frac{\partial t_1}{\partial x} = \frac{1}{y}$

$\Rightarrow t_2 = \frac{y}{z} \Rightarrow \frac{\partial t_2}{\partial x} = 0$

$\Rightarrow t_3 = \frac{z}{x} = \frac{\partial t_3}{\partial x} = -\frac{z}{x^2}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{1}{y} - \frac{\partial u}{\partial t_3}\frac{z}{x^2}$

To prove:- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

⇒ Multiplying by x Both side

$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial t_1} - \frac{z}{x}\frac{\partial u}{\partial t_3}$

Similary for y & z

$\Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial t_2} - \frac{x}{y}\frac{\partial u}{\partial t_1}$

$\Rightarrow z\frac{\partial u}{\partial z} = \frac{z}{x}\frac{\partial u}{\partial t_3} - \frac{y}{z}\frac{\partial u}{\partial t_2}$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \cancel{\frac{x}{y}\frac{\partial u}{\partial t_1}} - \cancel{\frac{z}{x}\frac{\partial u}{\partial t_3}}$
$+ \cancel{\frac{y}{z}\frac{\partial u}{\partial t_2}} - \cancel{\frac{x}{y}\frac{\partial u}{\partial t_1}} + \cancel{\frac{z}{x}\frac{\partial u}{\partial t_3}} - \cancel{\frac{y}{z}\frac{\partial u}{\partial t_2}}$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Hence proved

Ques3) If $u = f(r)$ where $r = \sqrt{x^2+y^2}$

PT $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Soln $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$

$\Rightarrow u = f(r) \Rightarrow \frac{\partial u}{\partial r} = f'(r)$

$\because r^2 = x^2+y^2 \Rightarrow \cancel{2}r\frac{\partial r}{\partial u} = \cancel{2}x \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

Similarly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$

$\Rightarrow \frac{\partial u}{\partial x} = f'(r)\frac{x}{r}$ again diff. w.r. to x

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \cancel{f'(x)}\ \cancel{\frac{x}{r}} + \frac{1}{r}f'(r) + x\frac{\partial}{\partial x}\left(\frac{f'(r)}{r}\right)$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + x\left[\frac{rf''(r) - f'(r)}{r^2}\right]\frac{\partial r}{\partial x}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + \frac{x^2}{r^3}(rf''(r) - f'(r))$

Similarly $\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + \frac{y^2}{r^3}(rf''(r) - f'(r))$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2f'(r)}{r} + \frac{x^2+y^2}{r^3}(rf''(r) - f'(r))$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2f'(r)}{r} + \frac{1}{\cancel{r}}\cancel{r}f''(r) - \frac{1}{r}f'(r)$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + f''(r)$

Ques4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \eta^2}$

where $x = r\cos\alpha - \eta\sin\alpha$

$y = r\sin\alpha + \eta\cos\alpha$

Soln $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$

$\Rightarrow \frac{\partial x}{\partial r} = \cos\alpha \Rightarrow \frac{\partial y}{\partial r} = \sin\alpha$

$\Rightarrow \frac{\partial u}{\partial r} = \cos\alpha\frac{\partial u}{\partial x} + \sin\alpha\frac{\partial u}{\partial y}$ again diff w.r. to r

$\Rightarrow \frac{\partial^2 u}{\partial r^2} = \cos^2\alpha\frac{\partial^2 u}{\partial x^2} + \sin^2\alpha\frac{\partial^2 u}{\partial y^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 u}{\partial x\partial y}$ — (1)

Similarly

$\Rightarrow \frac{\partial^2 u}{\partial \eta^2} = \sin^2\alpha\frac{\partial^2 u}{\partial x^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 u}{\partial y^2} + \cos^2\alpha\frac{\partial^2 u}{\partial r^2}$ — (2)

eqn (1) + eqn (2)

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial n^2} = (\sin^2\alpha + \cos^2\alpha)\frac{\partial^2 u}{\partial x^2} + (\sin^2\alpha + \cos^2\alpha)\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial n^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$ Hence proved

Summation of Series

$\Rightarrow$ Definite integral as a limit of sum
The value of the definite integral $\int_a^b f(x)dx$ is given as:

$$\Rightarrow \int_a^b f(x)\,dx = \lim_{\substack{h\to 0 \\ \text{or} \\ n\to\infty}} [h\{f(a) + f(a+h) + \dots + f(a+(\overline{n-1}h))\}] = \lim_{n\to\infty} h\left[\sum_{r=0}^{n-1} f(a+rh)\right]$$

Here $nh = b-a$ as $h\to 0$, $n\to\infty$, and $f(x)$ is continuous function of x in the given integral

r/n ki jagah x, $1/n$ ki jagah dx & $\sum_{r=0}^{n-1}$ ki jagah $\int_0^1$

Ques1) find the limit when $n\to\infty$ of the product
$$\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)+\dots+\cancel{\log}\left(1+\frac{n}{n}\right)\right\}^{1/n}$$

Soln $$A = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\dots+\cancel{\log}\left(1+\frac{n}{n}\right)\right]^{1/n}$$

taking log both side

$$\Rightarrow \log A = \lim_{n\to\infty}\frac{1}{n}\log\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\dots+\log\left(1+\frac{n}{n}\right)\right\}$$

$$\Rightarrow \log A = \frac{1}{n}\sum_{r=0}^{n}\log\left(1+\frac{r}{n}\right) \Rightarrow \log A = \int_0^1 \log(1+x).dx$$

$\Rightarrow$ ~~log A~~ = By ILET

$$\Rightarrow \log A = \log(1+x)\int_0^1 1\,dx - \int_0^1\left[\frac{d}{dx}\log(1+x)\int 1\times dx\right]dx$$

$$\Rightarrow \log A = [x\log(1+x)]_0^1 - \int_0^1 \frac{1}{1+x}\cdot x\,dx$$

$$\Rightarrow \log A = \log 2 - \int_0^1 \frac{1+x-1}{1+x}dx$$

$$\Rightarrow \log A = \log 2 - \int_0^1 \frac{1+x}{1+x}dx + \int_0^1 \frac{1}{1+x}dx$$

$$\Rightarrow \log A = \log 2 - [x]_0^1 + [\log(x+1)]_0^1$$

$$\Rightarrow \log A = \log 2 - 1 + \log 2$$

$$\Rightarrow \log A = \log 4 - 1 \Rightarrow \log A = \log 4 - \log e$$

$$\Rightarrow \boxed{A = e^{\log(4/e)}}$$ ans// or $\boxed{A = 4/e}$

Ques2) find the value: $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$

Soln $$A = \lim_{n\to\infty}\left[\frac{n!}{n}\right]^{1/n} = \lim_{n\to\infty}\left[\frac{n(n-1)(n-2)\dots 3\cdot 2\cdot 1}{n^n}\right]^{1/n}$$

$\Rightarrow$ Taking log both side

$$\Rightarrow \log A = \lim_{n\to\infty}\frac{1}{n}\log\left[\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\dots\frac{(n-1)}{n}\cdot\frac{n}{n}\right]$$

$$\Rightarrow \log A = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(\frac{r}{n}\right)$$

$$\Rightarrow \log A = \int_0^1 \log x \cdot 1\, dx$$

By ILET

$$\Rightarrow \log A = \log x \int_0^1 1\, dx - \int_0^1 \frac{d}{dx} \log x \int 1\, dx$$

$$\Rightarrow \log A = [x \log x]_0^1 - \int_0^1 \frac{1}{x} \cdot x\, dx$$

$$\Rightarrow \log A = \log 1 - [x]_0^1$$

$$\Rightarrow \log A = -1 \Rightarrow A = e^{-1} \Rightarrow \boxed{A = 1/e}$$

Ques3) find the value, as $n \to \infty$ of the series

$$\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3} \cdots \left(1+\frac{n}{n}\right)^{1/n}\right]$$

Soln $A = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3} \cdots \left(1+\frac{n}{n}\right)^{1/n}\right]$

Taking log both side

$$\Rightarrow \log A = \lim_{n\to\infty} \left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right)^{1/2} + \cdots + \log\left(1+\frac{n}{n}\right)^{1/n}\right]$$

$$\Rightarrow \log A = \lim_{n\to\infty} \sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right)^{1/r} \quad \{M \& D \text{ by } n\}$$

$$\Rightarrow \log A = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r/n} \log\left(1+\frac{r}{n}\right) \cdot \frac{1}{n}$$

$$\Rightarrow \log A = \int_0^1 \frac{1}{x} \log(1+x)\, dx$$

$$\Rightarrow \log A = \int_0^1 \frac{x - x^2/2 + x^3/3 - \cdots}{x}\, dx$$

$$\Rightarrow \log A = \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right) dx$$

$$\Rightarrow \log A = \left(x - \frac{x^2}{4} + \frac{x^3}{9}\right)_0^1$$

$$\Rightarrow \log A = 1 - \frac{1}{4} + \frac{1}{9} \cdots$$

$$\Rightarrow \log A = 1 - \frac{1}{2^2} + \frac{1}{3^2} \cdots$$

this series is fourier series & its value is $\pi^2/12$

$$\Rightarrow \log A = \pi^2/12$$

$$\Rightarrow A = e^{\pi^2/12}$$

Q) $\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \cdots \left(1+\frac{4n}{n}\right)\right]^{1/n} = \frac{5^5}{e^4}$

Soln $P = \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \cdots \left(1+\frac{4n}{n}\right)\right]^{1/n} = \frac{5^5}{e^4}$

taking log both side

$$\Rightarrow \log P = \lim_{n\to\infty} \frac{1}{n} \left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) \cdots \log\left(1+\frac{4n}{n}\right)\right]^{1/n} = 5^5/e^4$$

$$\Rightarrow \log P = \frac{1}{n} \sum_{r=1}^{4n} \log\left[1+\frac{r}{n}\right]$$

$$\Rightarrow \log P = \int_1^4 \log(1+x)\, dx$$

$$\Rightarrow \log P = [\log(1+x)]_0^4 \int_0^4 1\, dx - \int_0^4 \frac{d}{dx} \log(1+x) \int 1\, dx$$

$$\Rightarrow \log P = \log 5 [x]_0^4 - \int_0^4 \frac{1}{1+x} \times dx$$

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$\Rightarrow \log P = 4 \log 5 - \int_0^4 1 - \frac{1}{1+x}\, dx$

$\Rightarrow \log P = 4 \log 5 - \int_0^4 1\, dx + \int_0^4 \frac{1}{1+x}\, dx$

$\Rightarrow \log P = 4 \log 5 - [x]_0^4 + [\log(1+x)]_0^4$

$\Rightarrow \log P = 4 \log 5 - 4 + \log 5$

$\Rightarrow \log P = 5 \log 5 - 4$

$\Rightarrow \log P = \log 5^5 - 4$

$\Rightarrow \log \frac{P}{5^5} = -4 \Rightarrow \boxed{P = \frac{5^5}{e^4}}$ ans//

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